Biyani Girl's College

Concept Based Notes

B.Sc. Semester-IV

Paper: Real Analysis –II and Numerical Methods-II

Dr. Rachna Khandelwal

Department of Mathematics

BIYANI GIRLS COLLEGE

Think Tanks

Biyani Group of Colleges

BiyaniShikshanSamiti

Sector-3, Vidhyadhar Nagar,

Jaipur-302 023 (Rajasthan)

Ph :0141-2338371, 2338591-95

Fax 0141-2338007

E-mail:acad@biyanicolleges.org

Website www.gurukpo.com; www.biyanicolleges.org

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Preface

I am glad to present this book, especially designed to serve the needs of the students. The book has been written keeping in mind the general weakness in understanding the fundamental concepts of the topics. The book is self-explanatory and adopts the “Teach Yourself” style. It is based on question-answer pattern. The language of book is quite easy and understandable based on scientific approach.

Any further improvement in the contents of the book by making corrections, omission and inclusion is keen to be achieved based on suggestions from the readers for which the author shall be obliged.

I acknowledge special thanks to Mr. Rajeev Biyani, Chairman& Dr. Sanjay Biyani, Director (Acad.) Biyani Group of Colleges, who are the backbones and main concept provider and also have been constant source of motivation throughout this endeavor. They played an active role in coordinating the various stages of this endeavor and spearheaded the publishing work.

I look forward to receiving valuable suggestions from professors of various educational institutions, other faculty members and students for improvement of the quality of the book. The reader may feel free to send in their comments and suggestions to the under mentioned address.

Author

[UG0803-MAT-64T-203] - [Real Analysis-II & Numerical Analysis |

Unit-I : Properties of derivable functions, Darboux's and Rolle's theorem. Notation of limit, continuity and differentiability for functions of two variables. Directional derivative, total derivative, expression of total derivative in terms of partial derivatives. **(15 Lectures)**

Unit-II : Riemann integration Lower and Upper Riemann integrals, Riemann integrability, Mean value theorems of integral calculus, Fundamental theorem of integral calculus. Functions of bounded variations. **(15 Lectures)**

Unit-III : Differences, Relation between differences and derivatives, Differences of a polynomial, Newton's formulae for forward and backward interpolation, Divided differences. Newton's divided difference, Lagrange's interpolation formula, Numerical Differentiation, Derivatives from interpolation formulae. **(15 Lectures)**

Unit-IV : Numerical integration, Derivations of general quadrature formulas, Trapezoidal rule. Simpson's one-third, Simpson's three-eighth and Gauss's quadrature formulae. Numerical solution of Algebraic and Transcendental equations: Bisection method, secant method, Regula-Falsi method, Iteration method, Newton- Raphson Method. Numerical solutions of ordinary differential equations of first order with initial conditions using Euler and modified Euler's method. **(15 Lectures)**

Suggested Books and References -

1. Royden H, Fitzpatrick PM. Real analysis. China Machine Press; 2010.

2. Rudin W. Principles of mathematical analysis. New York: McGraw-hill: 1964.

3. Bartle RG, Sherbert DR. Introduction to real analysis. New York: Wiley; 2000.

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7. Iyengar SR, Jain RK. Numerical Methods. New Age International; 2009,

8. Sastry SS. Introductory methods of numerical analysis. PHI Learning Pvt. Ltd.; 2012.

UNIT-I

Q:1:

Show that $f(x) = |x-1|$ is not differentiable at $x = 0$

- The right-hand derivative at $x = 0$:

$$\lim_{x\to 1^+} \frac{|x-1|-0}{x-1} \lim_{x\to 1^+} \frac{x-1}{x-1} = 1$$

- The left-hand derivative at $x = 0$:

$$\lim_{x\to 1^-} \frac{|x-1|-0}{x-1} \lim_{x\to 1^+} \frac{-(x-1)}{x-1} = -1$$

So the right-hand and left-hand derivatives differ.

Let f be the scalar field defined on $\mathbb{R}^2$ as follows:

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that the above scalar field has directional derivative in every direction at $\mathbf{0}$ but which is not continuous at $\mathbf{0}$.

Q:2

Let $\mathbf{a} = (0,0)$ and let $\mathbf{y} = (a,b)$ be any vector. If $a \neq 0$ and $h \neq 0$ we have

$$\begin{aligned} \frac{f(\mathbf{a}+h\mathbf{y}) - f(\mathbf{a})}{h} &= \frac{f(\mathbf{0}+h\mathbf{y}) - f(\mathbf{0})}{h} \\ &= \frac{f(h\mathbf{y}) - f(\mathbf{0})}{h} \\ &= \frac{f(h(a,b)) - f(0,0)}{h} \\ &= \frac{f(h(a,b))}{h} \\ &= \frac{f(ha,hb)}{h} \\ &= \frac{1}{h}\left(\frac{(ha)(hb)^2}{(ha)^2+(hb)^4}\right) = \frac{ab^2}{a^2+h^2b^4}. \end{aligned}$$

$$\lim_{h\to 0} \frac{f(\mathbf{a}+h\mathbf{y})-f(\mathbf{a})}{h} = \lim_{h\to 0} \frac{ab^2}{a^2+h^2b^4}$$
$$\lim_{h\to 0} \frac{f(\mathbf{a}+h\mathbf{y})-f(\mathbf{a})}{h} = \frac{ab^2}{a^2+0.b^4}.$$
$$f'(\mathbf{0};\mathbf{y}) = \frac{b^2}{a}.$$

- If $\mathbf{y} = (0, b)$ we find, in a similar way, that $f'(\mathbf{0}; \mathbf{y}) = 0$.
- Therefore $f'(\mathbf{0}; \mathbf{y})$ exists for all directions $\mathbf{y}$.
- Also, $f(\mathbf{x}) \to 0$ as $\mathbf{x} \to \mathbf{0}$ along any straight line through the origin.
- However, at each point of the parabola $x = y^2$ (except at the origin) the function f has the value $1/2$.

Q:3 State and Proof Darboux Theorem .

Let us assume that $f'(a) < \lambda < f'(b)$ and consider a function $F : [a, b] \to R$ defined by $F(x) = f'(x) - \lambda x$. Then $F'(x) = f'(x) - \lambda$ and determine that $F'(a) = f'(a) - \lambda < 0$ and $F'(b) = f'(b) - \lambda > 0$.This implies that function F is monotonic on $[a, b]$ in the way there exist $x, y, z \in [a, b]$ such that $x < y < z$ satisfying only one the following condition (1) or (2).

1. $F(x) < F(y)$ and $F(y) > F(z)$. The theorem will be proved for this case.
2. $F(x) > F(y)$ and $F(y) < F(z)$. Proof in the case will follow a similar argument.

Suppose (a) holds, then we can observe following three cases

$F(x) < F(z)$

$F(z) < F(x)$

$F(z) = F(x)$

Case 1: Suppose that $F(x) > F(z)$. Then $F(x) < F(z) < F(y)$. F is differentiable on (a, b) so is f by hypothesis. Hence F is also continuous on (a, b). Hence, the intermediate value theorem applies to F on $[x, y] \subseteq (a, b)$ and we get $d \subseteq (x, y) \subseteq (a, b)$ such that $F(d) = F(z)$. Note that $a < x < d < y < z < b$.

Case 2: Suppose that $F(z) < F(x)$. Then $F(z) < F(x) < F(y)$. This theorem further follows by argument similar to that of case 1.

Case 3: Suppose that $F(z) = F(x)$. Then, we can finally apply rolle's theorem for F directly on $[x, z]$ and obtain $c(x, z) \subseteq (a, b)$ such that $F'(c) = 0$ leading to $f'(c) = \lambda$ as desired.

We arrive at the same conclusion if (2) holds by the same argument.

Darboux Problems and Solutions

Consider the function $f(x) = sin(x)$

1. What Values of the Derivatives of f are assured by Darboux Theorem on the Interval $[1, 3]$?

Solution:

As $f'(x) = cosx$, $f'(1) = 0.54030$ and $f'(3) = -0.98999$. Hence, the assured values are from -0.98999 to 0.54030.

Q:4 State and Proof Roll's Theorem .

Solution: Statement : Let f be continuous on [a, b] and differentiable on (a, b). Suppose f(a) = f(b). Then there exists c ∈ (a, b) such that f' (c) = 0.

Proof: If f is a constant function on [a, b] then, it follows from the definition of the derivative that f 0 (c) = 0 for all c ∈ [a, b]. Suppose there exists x ∈ (a, b) such that f(x) > f(a). (A similar argument can be given if f(x) < f(a)). Then by Theorem 5.3 there exists c ∈ (a, b) such that c is a point of maximum. Hence by Theorem 7.1, we have f' (c) = 0.

Theorem 7.1 (A necessary condition). Let f : [a, b] → R. Suppose x_0 ∈ (a, b) is a point of local maximum or local minimum. If f is differentiable at x0 then $f'(x_0) = 0$. Proof: Suppose x_0 ∈ (a, b) is a point of local maximum. Find (x_n) and (y_n) such that $x_n, y_n \in (a, b)$, $x_n < x_n < y_n$ for all n ∈ N and limn→∞ x_n = lim n→∞ $y_n = x_0$. Since x_0 is a point of local maximum, $f(x_n) - f(x_0) \leq 0$ and $f(y_n) - f(x_0) \leq 0$ for all n. Hence $f_0(x_0)$ = lim n→∞ $f(x_n) - f(x_0)$, $x_n - x_0 \geq 0$. Similarly, $f_0(x_0)$ = lim n→∞ $f(y_n) - f(x_0)$, $y_n - x_0 \leq 0$. Therefore $f_0(x_0) = 0$.

Q:5 Consider the equation $x^{13} + 7x^3 - 5 = 0$. To determine the number of solutions of this equations.

Solution: let $f(x) = x^{13} + 7x^3 - 5$. Then f(0) < 0 and f(1) > 0. By the IVT there is at least one positive root of f(x). Since that f(x) < 0 for x < 0, f(x) does not have a negative root. If there are two distinct positive roots of f(x), then by Rolle's theorem there is some x0 > 0 such that f 0 (x0) = 0 which is not true. Therefore f(x) = 0 has exactly one real solution.

Q:6. Consider the equation f(x) = 0 where $f(x) = x^8 + e^{-x} + 5x^2 - 2\cos x$.

Solution : Since $f_0(x) > 0$ for all x ∈ R, by Rolle's theorem, $f_0(x)$ has at most one real root. Again by Rolle's theorem, f(x) can have at most two real roots. Note that f(0) < 0, f(2) > 0 and f(−2) > 0. Hence, by the IVT f(x) = 0 has at least two real solutions. Hence f(x) = 0 has exactly two distinct real solutions. 3 (Uniqueness of the n-th root). Let α ∈ [0, ∞) and n ∈ N. It was shown, using the IVT, in Lecture 6 that the equation $x_n = \alpha$ has a solution in [0, ∞). We use Rolle's theorem to show that the equation x n = α has exactly one solution in [0, ∞). Let $f(x) = x_{n-\alpha}$ for x ∈ [0, ∞). Then $f_0(x) > 0$ for all (0, ∞). Therefore by Rolle's theorem, f(x) = 0 cannot have more than one real solution in [0, ∞)

Continuity of Function of Two Variables

Let $f(x, y)$ be a function of two independent variables x and y defined in some neighbourhood N of the point (a, b). Then, the function $f(x, y)$ is said to be **continuous** at a point (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

The continuity of $f(x, y)$ is also defined in the following way :

(i) A function $f(x, y)$ is said to be continuous at a point (a, b) if for $\varepsilon > 0$, there exists a neighbourhood N of (a, b) such that

$$|f(x, y) - f(a, b)| < \varepsilon \text{ for all } (x, y) \in N$$

Note 1.A function $f(x, y)$ is continuous at a point (a, b), then it requires that besides (a, b) the function $f(x, y)$ is defined in a certain nbd. of (a, b).

Note 2. The limit of the function $f(x, y)$ when $(x, y) \to (a, b)$ exists and equals the value $f(a, b)$.

Note 3.A function which is not continuous at a point is said to be discontinuous at that point.

Continuity of Function of Two Variables in a Domain

A function $f(x, y)$ is said to be **continuous in a domain** D, if it is continuous at every point of the domain D.

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$$

The function $f(x,y)$ is not continuous at origin, because the double limit $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

But $\lim_{x\to 0} f(x,0) = 0 = f(0,0)$ $\lim_{y\to 0} f(0,y) = 0 = f(0,0)$

Hence, $f(x,y)$ is continuous at (0,0), when considered as a function of a single variable x or that of y.

Algebra of Continuous Functions

If $f(x,y)$ and $g(x,y)$ are continuous functions of x and $y$at a point (a,b) for its domain of definition, then the functions

$f(x,y) + g(x,y)$,$f(x,y) - g(x,y)$,$f(x,y)g(x,y)$

$f(x,y)/g(x,y)$provided$g(a,b) \neq 0$are also continuous functions of x and y at (a,b).

Example 1. Prove that $f(x,y) = x^2 + 3y$ is continuous at the point $(1,2)$.

Solution : The value of the function $f(x,y) = x^2 + 3y$ at the point (1,2) is given by

$$f(1,2) = \frac{(x^2+3y)}{(1,2)} = (1)^2 + 3(2) = 1 + 6 = 7 \qquad \ldots(1)$$

Now, we have $|f(x,y) - f(1,2)| = |x^2 + 3y - 7|$

$$= |x^2 - 1 + 3y - 6|$$

$$= |(x+1)(x+1) + 3(y-2)|$$

$$\leq |x+1||x+1| + 3|y-2| \ldots(2)$$

Now, let $\delta > 0$ be such that $|x-1| < \delta$ and $|y-2| < \delta$ $\ldots(3)$

Choosing $\delta = 1$,we have$|x-1| < 1 \Rightarrow 0 < x < 2$

which also implies that $|x+1|<3 \Rightarrow |x+1|<3$ ...(4)

Thus, from (1), we have

$$|f(x,y)-f(1,2)|<3|x-1|+3|y-2|<3\delta+3\delta=6\delta \text{....(5)}$$

Again, take $\delta=\varepsilon/6$, then we get $|f(x,y)-f(1,2)|<\varepsilon$ ...(6)

∴ Require $\delta=\min\{1,\varepsilon/6\}$. ...(7)

Thus, for any $\varepsilon>0$, we can find $\delta=\min\{1,\varepsilon/6\}$ such that

$$|f(x,y)-f(1,2)|<\varepsilon \text{when} |x-1|<\delta \text{ and } |y-2|<\delta$$

∴ $\lim_{(x,y)\to(1,2)} f(x,y)=f(1,2)=7$ ...(8)

Hence, the given functionx^2+3y is continuous at the point (1,2).

Example 2. Show that function $f(x,y)=x-y$ is continuous for all $(x,y)\in R^2$.

Solution : Let $(a,b)\in R^2$, then we have $f(a,b)=a-b$ ...(1)

Now, we have $|f(x,y)-f(a,b)|=|(x-a)-(a-b)|$

$$=|(x-a)+(b-y)|$$

$$\leq |x-a|+|y-b|[\text{since}|x|=[-x]]\text{...(2)}$$

Now, let $\delta>0$ such that $|x-a|<\delta$ and $|y-b|<\delta$ ...(3)

By taking $\delta=\varepsilon/2$, we have $|f(x,y)-f(a,b)|<2\delta=\varepsilon$...(4)

Thus, for any $\varepsilon>0$, we can find a$\delta=\varepsilon/2$ such that

Now, let $\delta > 0$ such that $|x - a| < \delta$ and $|y - b| < \delta$ ...(3)

By taking $\delta = \varepsilon/2$, we have $|f(x,y) - f(a,b)| < 2\delta = \varepsilon$...(4)

Thus, for any $\varepsilon > 0$, we can find a$\delta = \varepsilon/2$ such that

$|f(x,y) - f(a,b)| < \varepsilon$, when$|x - a| < \delta$ and $|y - b| < \delta$...(5)

Hence, the function $f(x,y) = x - y$ is continuous at $(x,y) \in R^2$.

Again since (a,b) is an arbitrary point of R^2, therefore,

$f(x,y) = x - y$is continuous for all $(x,y) \in R^2$.

Problems: Directional Derivatives

The function $T = x^2 + 2y^2 + 2z^2$ gives the temperature at each point in space.

1. At the point $P = (1,1,1)$, in which direction should you go to get the most rapid decrease in T? What is the directional derivative in this direction?

Answer: We know that the fastest *increase* is in the direction of $\nabla T = \langle 2x, 4y, 4z \rangle$. At P, the fastest *decrease* is in the direction of $-\nabla T|_{(1,1,1)} = -\langle 1,2,2 \rangle$. The unit vector in this direction is $\hat{\mathbf{u}} = -\langle 1/3, 2/3, 2/3 \rangle$.

The rate of change in this direction is $-|\nabla T| = -3$. Equivalently, you could compute:

$$\left.\frac{dT}{ds}\right|_{P,\hat{\mathbf{u}}} = \nabla T|_P \cdot \hat{\mathbf{u}} = -3.$$

2. At P, about how far should you go in the direction found in part (1) to get a decrease of 0.3?

Answer: The directional derivative is a true derivative describing the limit of a ratio. In this case it equals $\lim_{\Delta s \to 0} \frac{\Delta T}{\Delta s}$, where Δs is the distance moved in the $\hat{\mathbf{u}}$ direction. Thus, we can write $\frac{\Delta T}{\Delta s} \approx \frac{dT}{ds}$.

In this problem we have $\frac{dT}{ds} = -3$ and $\Delta T = -0.3$.

$$\frac{-0.3}{\Delta s} \approx -3 \;\Rightarrow\; \Delta s \approx 0.1.$$

$$= \frac{m!}{(m-n)!} a^n (ax + \mathrm{b})^{m-n}$$

iii. $\boldsymbol{n^{th}}$ **Derivative of** $\boldsymbol{y = \log(ax + b)}$

Let $y = \log(ax + b)$

$$y_1 = \frac{a}{(ax+b)}$$

$$y_2 = \frac{-a^2}{(ax+b)^2}$$

$$y_3 = \frac{2!\, a^3}{(ax+b)^3}$$

$$\vdots$$

$$y_n = (-1)^{n-1} \frac{(n-1)!\, a^n}{(ax+b)^n}$$

iv. $\boldsymbol{n^{th}}$ **Derivative of** $\boldsymbol{y = \sin(ax + b)}$

Let $y = \sin(ax + b)$

$$y_1 = a\cos(ax + b) = a\sin\left(ax + b + \frac{\pi}{2}\right)$$

$$y_2 = a^2\cos\left(ax + b + \frac{\pi}{2}\right) = a^2\sin\left(ax + b + \frac{2\pi}{2}\right)$$

$$\vdots$$

$$y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$$

Similarly if $\boldsymbol{y = \cos(ax + b)}$

$$y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$$

Example 7 Find the n^{th} derivative of $\tan^{-1}\frac{x}{a}$

Solution: Let $y = \tan^{-1}\frac{x}{a}$

$$\Rightarrow y_1 = \frac{dy}{dx} = \frac{1}{a\left(1+\frac{x^2}{a^2}\right)} = \frac{a}{x^2+a^2} = \frac{a}{x^2-(ai)^2}$$

$$= \frac{a}{(x+ai)(x-ai)} = \frac{a}{2ai}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)$$

$$= \frac{1}{2i}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)$$

Differentiating above $(n-1)$ times w.r.t. x, we get

$$y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-ai)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+ai)^n}\right]$$

Substituting $x = r\,cos\theta,\ a = r\,sin\theta$ such that $\theta = \tan^{-1}\frac{x}{a}$

$$\Rightarrow y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{r^n(\cos\theta - i\,sin\theta)^n} - \frac{1}{r^n(\cos\theta + isin\theta)^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2ir^n}\left[(\cos\theta - i\,sin\theta)^{-n} - (\cos\theta + i\,sin\theta)^{-n}\right]$$

Using De Moivre's theorem, we get

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2ir^n}[\cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta]$$

$$= \frac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta$$

$$= \frac{(-1)^{n-1}(n-1)!}{\left(\frac{a}{\sin\theta}\right)^n}\sin n\theta \qquad \because a = r\,sin\theta$$

$$= \frac{(-1)^{n-1}(n-1)!}{a^n}\sin n\theta\ \sin^n\theta \qquad \text{where } \theta = \tan^{-1}\frac{a}{x}$$

i. n^{th} Derivative of e^{ax}

Let $y = e^{ax}$

$y_1 = ae^{ax}$

$y_2 = a^2e^{ax}$

$\vdots$

$y_n = a^n e^{ax}$

ii. n^{th} Derivative of $(ax+b)^m$, m is a +ve integer greater than n

Let $y = (ax+b)^m$

$y_1 = m\,a(ax+b)^{m-1}$

$y_2 = m(m-1)a^2(ax+b)^{m-2}$

$\vdots$

$y_n = m(m-1)\dots(m-n+1)a^n(ax+b)^{m-n}$

Lemma 1.1. (Refinement) *Let $f : [a, b] \to \mathbb{R}$ be bounded. Let P be a partition of $[a, b]$ and P^* be a refinement of P. Then*

$$L(f, P) \leq L(f, P^*) \leq U(f, P^*) \leq U(f, P)\,. \tag{8}$$

Proof. It follows from (2), (7), and (6) that

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{n} \underline{m}_i\,(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} L(f, P_i^*) = L(f, P^*) \\ &\leq U(f, P^*) = \sum_{i=1}^{n} U(f, P_i^*) \\ &\leq \sum_{i=1}^{n} \overline{m}_i\,(x_i - x_{i-1}) = U(f, P)\,. \end{aligned}$$

1.3. **Comparisons.** A key step in our development will be to develop comparisons of $L(f, P^1)$ and $U(f, P^2)$ for any two partitions P^1 and P^2, of $[a, b]$.

Definition 1.4. *Given any two partitions, P^1 and P^2, of $[a, b]$ we define $P^1 \vee P^2$ to be the partition whose set of partition points is the union of the partition points of P^1 and the partition points of P^2. We call $P^1 \vee P^2$ the* supremum *of P^1 and P^2.*

It is easy to argue that $P^1 \vee P^2$ is the smallest partition of $[a, b]$ that is a refinement of both P^1 and P^2. It is therefore sometimes called the *smallest common refinement* of P^1 and P^2.

Lemma 1.2. (Comparison) *Let $f : [a, b] \to \mathbb{R}$ be bounded. Let P^1 and P^2 be partitions of $[a, b]$. Then*

$$L(f, P^1) \leq U(f, P^2)\,. \tag{9}$$

Proof. Because $P^1 \vee P^2$ is a refinement of both P^1 and P^2, it follows from the Refinement Lemma that

$$L(f, P^1) \leq L(f, P^1 \vee P^2) \leq U(f, P^1 \vee P^2) \leq U(f, P^2)\,.$$

□

Because the partitions P^1 and P^2 on either side of inequality (9) are independent, we may obtain sharper bounds by taking the supremum over P^1 on the left-hand side, or the infimum over P^2 on the right-hand side. Indeed, we prove the following.

Lemma 1.2. (Comparison) *Let $f : [a, b] \to \mathbb{R}$ be bounded. Let P^1 and P^2 be partitions of $[a, b]$. Then*

$$L(f, P^1) \le U(f, P^2) \,. \tag{9}$$

Proof. Because $P^1 \vee P^2$ is a refinement of both P^1 and P^2, it follows from the Refinement Lemma that

$$L(f, P^1) \le L(f, P^1 \vee P^2) \le U(f, P^1 \vee P^2) \le U(f, P^2) \,.$$

□

Because the partitions P^1 and P^2 on either side of inequality (9) are independent, we may obtain sharper bounds by taking the supremum over P^1 on the left-hand side, or the infimum over P^2 on the right-hand side. Indeed, we prove the following.

Lemma 1.3. (Sharp Comparison) *Let $f : [a, b] \to \mathbb{R}$ be bounded. Let*

$$\begin{aligned} \overline{L}(f) &= \sup \big\{ L(f, P) \,:\, P \text{ is a partition of } [a, b] \big\} \,, \\ \underline{U}(f) &= \inf \big\{ U(f, P) \,:\, P \text{ is a partition of } [a, b] \big\} \,. \end{aligned} \tag{10}$$

Let P^1 and P^2 be partitions of $[a, b]$. Then

$$L(f, P^1) \le \overline{L}(f) \le \underline{U}(f) \le U(f, P^2) \,. \tag{11}$$

Moreover, if

$$L(f, P) \le A \le U(f, P) \quad \text{for every partition } P \text{ of } [a, b] \,,$$

then $A \in [\overline{L}(f), \underline{U}(f)]$.

Remark. Because it is clear from (10) that $\overline{L}(f)$ and $\underline{U}(f)$ depend on $[a, b]$, strictly speaking these quantities should be denoted $\overline{L}(f, [a, b])$ and $\underline{U}(f, [a, b])$. This would be necessary if more than one interval was involved in the discussion. However, that is not the case here. We therefore embrace the less cluttered notation.

Question 1 (2018-19 Final Q2). Define a function $g : [0, \pi/2] \to \mathbb{R}$ by

$$g(x) = \begin{cases} \cos^2 x, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{otherwise.} \end{cases}$$

Find the upper and lower Riemann integrals of g over $[0, \pi/2]$. Is it Riemann integrable?

Solution. Let's find the lower and upper integrals of g.

- **Lower integral**: Let P be any partition of $[0, \pi/2]$. Note that each subinterval $[x_{i-1}, x_i]$ must containing some irrational number, so

  $$m_i(g, P) = 0, \quad \forall i = 1, ..., n.$$

  It follows that the lower sum is given by

  $$L(g, P) = \sum_{i=1}^{n} m_i(g, P)\Delta x_i = 0.$$

  Taking infimum over all partition P, the lower integral of g is given by

  $$\underline{\int_0^{\pi/2}} g = 0.$$

- **Upper integral**: Let P be any partition of $[0, \pi/2]$. Note that $\cos^2 x$ is decreasing on each subinterval $[x_{i-1}, x_i]$ and rational numbers are dense, so

  $$M_i(g, P) = \cos^2(x_{i-1}), \quad \forall i = 1, ..., n.$$

  Consider $f : [0, \pi/2] \to \mathbb{R}$ defined by $f(x) = \cos^2 x$. We also have

  $$M_i(f, P) = \cos^2(x_{i-1}), \quad \forall i = 1, ..., n.$$

  It follows that

  $$U(g, P) = \sum_{i=1}^{n} M_i(g, P)\Delta x_i = \sum_{i=1}^{n} M_i(f, P)\Delta x_i = U(f, P).$$

  Since f and g have the same upper sum over arbitrary partitions of $[0, \pi/2]$, they have

Question 4 (2016-17 Final Q2). Let f be a function defined by

$$f(x) = \frac{\sin x}{x}, \quad \text{for } x \geq 1.$$

(i) Show that the integral $\int_1^\infty f(x)dx$ is convergent.

(ii) Show that the integral $\int_1^\infty |f(x)|dx$ is divergent.

Solution. .

(i) Fix $T > 1$. Note that by **Integration by Parts**,

$$\int_1^T f(x)dx = \int_1^T \frac{\sin x}{x}dx = \left[-\frac{\cos x}{x}\right]_1^T - \int_1^T \frac{\cos x}{x^2}dx.$$

Hence it suffices to show that the improper integral $\int_1^\infty \frac{\cos x}{x^2}dx$ converges.

Note that for any $A_2 > A_1 > 1$, we have

$$\left|\int_{A_1}^{A_2} \frac{\cos x}{x^2}dx\right| \leq \int_{A_1}^{A_2} \frac{|\cos x|}{x^2}dx \leq \int_{A_1}^{A_2} \frac{1}{x^2}dx = \frac{1}{A_1} - \frac{1}{A_2} \leq \frac{1}{A_1}.$$

Hence for any $\varepsilon > 0$, we can choose $M > 1$ such that $1/M < \varepsilon$. Then whenever $A_2 > A_1 > M$,

$$\left|\int_{A_1}^{A_2} \frac{\cos x}{x^2}dx\right| \leq \frac{1}{A_1} < \frac{1}{M} < \varepsilon.$$

It follows by **Cauchy Criterion** that $\int_1^\infty \frac{\cos x}{x^2}dx$ converges.

(ii) Note that since $|f(x)|$ is non-negative, we have

$$\int_1^\infty |f(x)|dx \geq \int_\pi^{(N+1)\pi} \frac{|\sin x|}{x}dx = \sum_{k=1}^{N} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}dx.$$

For each $k \in \mathbb{N}$, we can substitute $x = t + k\pi$, then

$$\int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}dx = \int_0^\pi \frac{|\sin(t + k\pi)|}{t + k\pi}dt = \int_0^\pi \frac{\sin t}{t + k\pi}dt \geq \frac{1}{(k+1)\pi}\int_0^\pi \sin t dt.$$

(ii) Note that since $|f(x)|$ is non-negative, we have

$$\int_1^\infty |f(x)|dx \geq \int_\pi^{(N+1)\pi} \frac{|\sin x|}{x}dx = \sum_{k=1}^N \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}dx.$$

For each $k \in \mathbb{N}$, we can substitute $x = t + k\pi$, then

$$\int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}dx = \int_0^\pi \frac{|\sin(t+k\pi)|}{t+k\pi}dt = \int_0^\pi \frac{\sin t}{t+k\pi}dt \geq \frac{1}{(k+1)\pi}\int_0^\pi \sin t dt.$$

Write $A = \displaystyle\int_0^\pi \sin t dt$, we have

$$\int_1^\infty |f(x)|dx \geq \sum_{k=1}^N \frac{A}{(k+1)\pi} = \frac{A}{\pi}\sum_{k=1}^N \frac{1}{k+1}.$$

Since the above inequality holds for all $N \in \mathbb{N}$ and the harmonic series diverges to ∞, it follows that $\displaystyle\int_1^\infty |f(x)dx$ is also divergent.

Question 5 (2018-19 Final Q3). Let f be a continuous function on $[a,b]$ and $\varphi : [\alpha, \beta] \to \mathbb{R}$ be continuously differentiable such that $\varphi(\alpha) = a$ and $\varphi(\beta) = b$. Show that

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$$

(Hint: Consider the functions $F(u) = \int_a^u f(x)dx$ and $H(t) = F(\varphi(t))$.)

Solution. Define the functions $F : [a,b] \to \mathbb{R}$ and $H : [\alpha, \beta] \to \mathbb{R}$ by

$$F(u) = \int_a^u f(x)dx \quad \text{and} \quad H(t) = F(\varphi(t)).$$

Since f is continuous, we have $F' = f$ by the **Fundamental Theorem of Calculus**. Also, by **Chain Rule**, we have $H'(t) = F'(\varphi(t))\varphi'(t) = f(\varphi(t))\varphi'(t)$. It follows again by the **Fundamental Theorem of Calculus** that

$$\int_a^b f(x)dx = F(b) - F(a) = H(\beta) - H(\alpha) = \int_\alpha^\beta H'(t)dt = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$$

Theorem 10.2. (Continuous Integrability) *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is Riemann integrable over $[a, b]$.*

Remark. The fact that f is continuous over $[a, b]$ implies that it is bounded over $[a, b]$ and that it is uniformly continuous over $[a, b]$. The fact f is bounded is needed to know that the Darboux sums $L(f, P)$ and $U(f, P)$ make sense for any partition P. The fact f is uniformly continuous will play the central role in our proof.

Proof. Let $\epsilon > 0$. Because f is uniformly continuous over $[a, b]$, there exists a $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{b - a} \quad \text{for every } x, y \in [a, b] .$$

Let $P = [x_0, x_1, \cdots, x_n]$ be any partition of $[a, b]$ such that $|P| < \delta$. Because f is continuous, it takes on extreme values over each subinterval $[x_{i-1}, x_i]$ of P. Hence, for every $i = 1, \cdots, n$ there exist points $\overline{x}_i$ and $\underline{x}_i$ in $[x_{i-1}, x_i]$ such that $\overline{m}_i = f(\overline{x}_i)$ and $\underline{m}_i = f(\underline{x}_i)$. Because $|P| < \delta$ it follows that $|\overline{x}_i - \underline{x}_i| < \delta$, whereby

$$\overline{m}_i - \underline{m}_i = f(\overline{x}_i) - f(\underline{x}_i) < \frac{\epsilon}{b - a} .$$

We thereby obtain

$$\begin{aligned} 0 \le U(f, P) - L(f, P) &= \sum_{i=1}^{n} (\overline{m}_i - \underline{m}_i)\,(x_i - x_{i-1}) \\ &\le \frac{\epsilon}{b - a} \sum_{i=1}^{n} (x_i - x_{i-1}) = \frac{\epsilon}{b - a}\,(b - a) = \epsilon . \end{aligned}$$

Corollary 11.1. *Let $f : [a, b] \to \mathbb{R}$ have a primitive over $[a, b]$. Let $x_o \in [a, b]$ and $y_o \in \mathbb{R}$. Then f has a unique primitive F such that $F(x_o) = y_o$.*

Proof. Exercise. □

Exercise. Let $f : [0, 3] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{for } 0 \le x < 1 , \\ -x & \text{for } 1 \le x < 2 , \\ 1 & \text{for } 2 \le x \le 3 . \end{cases}$$

Find F, the primitive of f over $[0, 3]$ specified by $F(0) = 1$.

We are now ready to for the big theorem.

Theorem 11.1. (First Fundamental Theorem of Calculus) *Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and have a primitive F over $[a,b]$. Then*

$$\int_a^b f = F(b) - F(a)\,.$$

Remark. This theorem recasts the problem of evaluating definite integrals to that of finding an explicit primitive of f. While such an explicit primitive cannot always be found, it can be found for a wide class of elementary integrands f.

Proof. By characterization (3) of the Riemann-Darboux Theorem, the result will follow if for every partition P of $[a,b]$ we have

$$L(f,P) \leq F(b) - F(a) \leq U(f,P)\,. \tag{11.2}$$

Let P be an arbitrary partition of $[a,b]$. Let $[p_0, \cdots, p_n]$ be the partition of $[a,b]$ associated with the primitive F. Let $P^* = P \vee [p_0, \cdots, p_n]$. Express P^* in terms of its partition points as $P^* = [x_0, \cdots, x_{n^*}]$. Then for every $i = 1, \cdots, n^*$ we know that $F : [x_{i-1}, x_i] \to \mathbb{R}$ is continuous, and that $F : (x_{i-1}, x_i) \to \mathbb{R}$ is differentiable. Then by the Lagrange Mean-Value Theorem there exists $q_i \in (x_{i-1}, x_i)$ such that

$$F(x_i) - F(x_{i-1}) = F'(q_i)\,(x_i - x_{i-1}) = f(q_i)\,(x_i - x_{i-1})\,.$$

Because $\underline{m}_i \leq f(q_i) \leq \overline{m}_i$, we see from the above that

$$\underline{m}_i\,(x_i - x_{i-1}) \leq F(x_i) - F(x_{i-1}) \leq \overline{m}_i\,(x_i - x_{i-1}) \quad \text{for every } i = 1, \cdots, n^*\,.$$

Upon summing these inequalities we obtain

$$L(f,P^*) \leq \sum_{i=1}^{n^*} \big(F(x_i) - F(x_{i-1})\big) \leq U(f,P^*)\,.$$

Corollary 11.2. *Let $F : [a,b] \to \mathbb{R}$ be continuous over $[a,b]$ and differentiable over (a,b). Suppose $F' : (a,b) \to \mathbb{R}$ is bounded over (a,b) and Riemann integrable over every $[c,d] \subset (a,b)$. Let f be any extension of F' to $[a,b]$. Then f is Riemann integrable over $[a,b]$, F is a primitive of f over $[a,b]$, and*

$$\int_a^b f = F(b) - F(a)\,.$$

Example. Let F be defined over $[-1,1]$ by

$$F(x) = \begin{cases} x\cos(\log(1/|x|)) & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Then F is continuous over $[-1,1]$ and continuously differentiable over $[-1,0) \cup (0,1]$ with

$$F'(x) = \cos(\log(1/|x|)) + \sin(\log(1/|x|))\,.$$

As this function is bounded, we have

$$\int_{-1}^{1} \big[\cos(\log(1/|x|)) + \sin(\log(1/|x|))\big]\,\mathrm{d}x = F(1) - F(-1) = 2\,.$$

Here the integrand can be assigned any value at $x = 0$. We first apply the above corollary to $[-1,0]$ and to $[0,1]$, and then use interval additivity.

Consider $f : [0, \pi/2] \to \mathbb{R}$ defined by $f(x) = \cos^2 x$. We also have

$$M_i(f, P) = \cos^2(x_{i-1}), \quad \forall i = 1, ..., n.$$

It follows that

$$U(g, P) = \sum_{i=1}^{n} M_i(g, P)\Delta x_i = \sum_{i=1}^{n} M_i(f, P)\Delta x_i = U(f, P).$$

Since f and g have the same upper sum over arbitrary partitions of $[0, \pi/2]$, they h
the same upper integral, hence

$$\overline{\int_0^{\pi/2}} g = \overline{\int_0^{\pi/2}} f = \int_0^{\pi/2} f = \int_0^{\pi/2} \cos^2 x dx = \int_0^{\pi/2} \frac{1 + \cos 2x}{2} dx = \frac{\pi}{4}.$$

In summary, the lower and upper intgral of g is given by

$$\underline{\int_0^{\pi/2}} g = 0 \quad \text{and} \quad \overline{\int_0^{\pi/2}} g = \frac{\pi}{4}.$$

Since they are unequal, g is **not** Riemann integrable.

11.3. **Integration by Parts.** An important consequence of the First Fundamental Theorem of Calculus and the Product Rule for derivatives is the following lemma regarding integration by parts.

Proposition 11.1. (Integration by Parts) *Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be Riemann integrable and have primitives F and G respectively over $[a, b]$. Then Fg and Gf are Riemann integrable over $[a, b]$ and*

$$\int_a^b Fg = F(b)G(b) - F(a)G(a) - \int_a^b Gf\,. \tag{11.3}$$

Proof. The functions F and G are Riemann integrable over $[a, b]$ because they are continuous. The functions Fg and Gf are therefore Riemann integrable over $[a, b]$ by the Product Lemma. Suppose we know that $FG : [a, b] \to \mathbb{R}$ is a primitive of $Fg + Gf$ over $[a, b]$. Then equation (11.3) follows from the First Fundamental Theorem of Calculus and the Additivity Lemma.

All that remains to be shown is that FG is a primitive of $Fg+Gf$ over $[a, b]$. It is clear that FG is continuous over $[a, b]$ because it is the product of continuous functions. Now let P and Q be the partitions of $[a, b]$ associated with the primitives F and G respectively. Let $R = P \vee Q$. Express R in terms of its partition points as $R = [r_0, \cdots, r_n]$. Then for every $i = 1, \cdots, n$ the function FG is differentiable over (r_{i-1}, r_i) with (by the Product Rule)

$$\begin{aligned}(FG)'(x) = F(x)G'(x) + G(x)F'(x) \quad &= F(x)g(x) + G(x)f(x) \\ = (Fg + Gf)(x) \qquad\qquad &\text{for every } x \in (r_{i-1}, r_i)\,.\end{aligned}$$

Therefore FG is a primitive of $Fg + Gf$ over $[a, b]$. □

In the case where f and g are continuous over $[a, b]$ then the Second Fundamental Theorem of Calculus implies that f and g have primitives F and G that are continuously differentiable over $[a, b]$. In that case integration by parts reduces to the following.

Corollary 11.3. *Let $F : [a, b] \to \mathbb{R}$ and $G : [a, b] \to \mathbb{R}$ be continuously differentiable over $[a, b]$. Then*

$$\int_a^b FG' = F(b)G(b) - F(a)G(a) - \int_a^b GF'\,.$$

11.4. **Substitution.** An important consequence of the First Fundamental Theorem of Calculus and the Chain Rule for derivatives is the following proposition regarding changing the variable of integration in a definite integral by monotonic substitution $y = G(x)$.

UNIT-II

11.5. **Integral Mean-Value Theorem.** We will now give a theorem that a first glance may not seem to have a connection either with the Fundamental Theorems of Calculus or with a Mean-Value Theorem for differentiable functions. However, we will see there is a connection.

Theorem 11.3. (Integral Mean-Value) *Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $g : [a, b] \to \mathbb{R}$ be Riemann integrable and positive almost everywhere over $[a, b]$. Then there exists a point $p \in (a, b)$ such that*

$$\int_a^b fg = f(p) \int_a^b g \,. \tag{11.5}$$

Remark. The connection of this theorem to both the First and Second Fundamental Theorem of Calculus and to the Cauchy Mean-Value Theorems for differentiable functions is seen when both f and g are continuous. Then by the Second Fundamental Theorem of Calculus fg and g have continuously differentiable primitives F and G. The Cauchy Mean-Value Theorem applied to F and G then yields a $p \in (a, b)$ such that

$$F(b) - F(a) = \frac{F'(p)}{G'(p)} \big(G(b) - G(a)\big) = f(p)\big(G(b) - G(a)\big) \,.$$

By the First Fundamental Theorem of Calculus we therefore have

$$\int_a^b fg = F(b) - F(a) = f(p) \big(G(b) - G(a)\big) = f(p) \int_a^b g \,.$$

In other words, when both f and g are continuous the Integral Mean-Value Theorem is just the Cauchy Mean-Value Theorem for differentiable functions applied to primitives of fg and g.

Remark. A simpler version of the Integral Mean-Value Theorem only considers the case $g(x) = 1$. In that case, if $f : [a, b] \to \mathbb{R}$ is continuous, there exists a point $p \in (a, b)$ such that

$$f(p) = \frac{1}{b - a} \int_a^b f \,.$$

This is proved by simply applying the Lagrange Mean-Value Theorem to a primitive of f. If we interpret the right-hand side above as the average of f over the interval $[a, b]$ then the theorem asserts that f takes on its average value.

$$= \left(\frac{\varepsilon}{b-a}\right) \sum_{i=1}^{n} (x_i - x_{i-1}) = \varepsilon$$

Therefore it follows from the Squeeze Theorem that $f \in \mathcal{R}[a,b]$.

Theorem 3.2.8. *If $f : [a,b] \to \mathbb{R}$ is monotone on $[a,b]$, then $f \in \mathcal{R}[a,b]$.*

Proof. Assume that f is increasing on $I = [a,b]$. Partitioning the interval into n equal subintervals $I_k = [x_{k-1}, x_k]$ gives us $x_k - x_{k-1} = (b-a)/n, k = 1,2,\ldots,n$. Since f is increasing on I_k, its minimum value is attained at the left endpoint x_{k-1} and its maximum value is attained at the right endpoint x_k. Therefore, we define the step functions
$\alpha(x) := f(x_{k-1})$ and $\omega(x) := f(x_k)$ for $x \in [x_{k-1}, x_k), k = 1,2,\ldots,n-1$, and
$\alpha(x) := f(x_{n-1})$ and $\omega(x) := f(x_n)$ for $x \in [x_{n-1}, x_n]$.
Then we have $\alpha(x) \leq f(x) \leq \omega(x)$ for all $x \in I$, and

$$\int_a^b \alpha = \frac{b-a}{n}\left(f(x_0) + f(x_1) + \cdots + f(x_{n-1})\right)$$
$$\int_a^b \omega = \frac{b-a}{n}\left(f(x_1) + \cdots + f(x_{n-1}) + f(x_n)\right).$$

Subtracting, and noting the many cancellations, we obtain

$$\int_a^b (\omega - \alpha) = \frac{b-a}{n}\left(f(x_n) - f(x_0)\right) = \frac{b-a}{n}(f(b) - f(a)).$$

Thus for a given $\varepsilon > 0$, we choose n such that $n > (b-a)(f(b) - f(a))/\varepsilon$. Then we have $\int_a^b (\omega - \alpha) < \varepsilon$ and the Squeeze Theorem implies that f is integrable on I. □

Theorem 3.2.7. *$f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then $f \in \mathcal{R}[a, b]$.*

Proof. Using the fact that a real valued continuous function on a closed and bounded interval uniformly continuous we get f is uniformly continuous on $[a, b]$. Therefore, given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $u, v \in [a, b]$ and $|u - v| < \delta_\varepsilon$, then we have $|f(u) - f(v)| < \varepsilon/(b - a)$.

Let $\mathcal{P} = \{I_i\}_{i=1}^n$ be a partition such that $\|\mathcal{P}\| < \delta_\varepsilon$. Applying Maximum-Minimum Theorem 2.2.7 we let $u_i \in I_i$ be a point where f attains its minimum value on I_i, and let $v_i \in I_i$ be a point where f attains its maximum value on I_i.
Let α_ε be the step function defined by
$\alpha_\varepsilon(x) := f(u_i)$ for $x \in [x_{i-1}, x_i)$ $(i = 1, \ldots, n-1)$ and
$\alpha_\varepsilon(x) := f(u_n)$ for $x \in [x_{n-1}, x_n]$.
Let ω_ε be defined similarly using the points v_i instead of the u_i.
$\omega_\varepsilon(x) := f(v_i)$ for $x \in [x_{i-1}, x_i)$ $(i = 1, \ldots, n-1)$ and
$\omega_\varepsilon(x) := f(v_n)$ for $x \in [x_{n-1}, x_n]$.
Then one has

$$\alpha_\varepsilon(x) \le f(x) \le \omega_\varepsilon(x) \quad \text{for all} \quad x \in [a, b].$$

Moreover, it is clear that

$$\begin{aligned} 0 \le \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) &= \sum_{i=1}^n (f(v_i) - f(u_i))(x_i - x_{i-1}) \\ &< \sum_{i=1}^n \left(\frac{\varepsilon}{b-a}\right)(x_i - x_{i-1}) \end{aligned}$$

Theorem : Fundamental Theorem of Calculus (First Form) :Suppose there is a finite set E in [a, b] and functions f, f :[a, b] → R such that:

(a) F is continuous on [a, b],

(b) $F'(x) = f(x)$ *for all* $x \in [a, b] \setminus E$,

(c) f *belongs to* $\mathcal{R}[a, b]$.

Then we have

$$\int_a^b f = F(b) - F(a). \tag{3.13}$$

Proof. We will prove the theorem in the case where $E := \{a, b\}$. The general case can be obtained by breaking the interval into the union of a finite number of intervals.

Let $\varepsilon > 0$ be given. Since $f \in \mathcal{R}[a, b]$ by assumption (c), there exists $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition with $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$, then

$$\left| S(f; \dot{\mathcal{P}}) - \int_a^b f \right| < \varepsilon. \tag{3.14}$$

If the subintervals in $\dot{\mathcal{P}}$ are $[x_{i-1}, x_i]$, then the Mean Value Theorem 1.5.5 applied to F on $[x_{i-1}, x_i]$ implies that there exists $u_i \in (x_{i-1}, x_i)$ such that

$$F(x_i) - F(x_{i-1}) = F'(u_i) \cdot (x_i - x_{i-1}) \quad \text{for} \quad i = 1, \ldots, n.$$

If we add these terms, note the telescoping of the sum, and use the fact that
$F'(u_i) = f(u_i)$, we obtain

$$F(b)-F(a)=\sum_{i=1}^{n}\left(F\left(x_i\right)-F\left(x_{i-1}\right)\right)=\sum_{i=1}^{n} f\left(u_i\right)\left(x_i-x_{i-1}\right)$$

Now let $\dot{\mathcal{P}}_u:=\{([x_{i-1},x_i],u_i)\}_{i=1}^{n}$, so the sum on the right equals $S\left(f;\dot{\mathcal{P}}_u\right)$. If we substitute $F(b)-F(a)=S\left(f;\dot{\mathcal{P}}_u\right)$ into (3.14), we conclude that

$$\left|F(b)-F(a)-\int_a^b f\right|<\varepsilon$$

But, since $\varepsilon>0$ is arbitrary, we infer that equation (3.13) holds.

Q:1

(a) If $F(x):=\frac{1}{2}x^2$ for all $x\in[a,b]$, then $F'(x)=x$ for all $x\in[a,b]$. Further, $f=F'$ is continuous so it is in $\mathcal{R}[a,b]$. Therefore the Fundamental Theorem (with $E=\emptyset$) implies that

$$\int_a^b x\,dx=F(b)-F(a)=\frac{1}{2}\left(b^2-a^2\right).$$

Q:2 (b) If G(x) := Arctan x for x ∈ [a, b], then $G_0(x) = 1/(x^2 + 1)$ for a

$x\in[a,b]$; also G' is continuous, so it is in $\mathcal{R}[a,b]$. Therefore the Fundamental Theorem (with $E=\emptyset$) implies that

$$\int_a^b \frac{1}{x^2+1}dx=\operatorname{Arctan} b-\operatorname{Arctan} a$$

Q:3

(d) If $H(x):=2\sqrt{x}$ for $x\in[0,b]$, then H is continuous on $[0,b]$ and $H'(x)=1/\sqrt{x}$ for $x\in(0,b]$. Since $h:=H'$ is not bounded on $(0,b]$, it does not belong to $\mathcal{R}[0,b]$ no matter how we define $h(0)$. Therefore, the Fundamental Theorem 3.3.1 does not apply.

Theorem 3.3.4. Fundamental Theorem of Calculus
(**Second Form**) *Let $f\in\mathcal{R}[a,b]$ and let f be continuous at a point $c\in[a,b]$. Then the indefinite integral, defined by (3.15), is differentiable at c and $F'(c)=f(c)$.*

Proof. We will suppose that $c\in[a,b)$ and consider the right-hand derivative of F at c. Since f is continuous at c, given $\varepsilon>0$ there exists $\eta_\varepsilon>0$

such that if $c \leq x < c + \eta_\varepsilon$, then

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon. \tag{3.16}$$

Let h satisfy $0 < h < \eta_\varepsilon$. The Additivity Theorem 3.2.9 implies that f is integrable on the intervals $[a, c], [a, c+h]$ and $[c, c+h]$ and that

$$F(c+h) - F(c) = \int_c^{c+h} f$$

Now on the interval $[c, c+h]$ the function f satisfies inequality (3.16), so that we have

$$(f(c) - \varepsilon) \cdot h \leq F(c+h) - F(c) = \int_c^{c+h} f \leq (f(c) + \varepsilon) \cdot h$$

If we divide by $h > 0$ and subtract $f(c)$, we obtain

$$\left| \frac{F(c+h) - F(c)}{h} - f(c) \right| \leq \varepsilon.$$

But, since $\varepsilon > 0$ is arbitrary, we conclude that the right-hand limit is given by

$$\lim_{x \to 0+} \frac{F(c+h) - F(c)}{h} = f(c)$$

It is proved in the same way that the left-hand limit of this difference quotient also equals $f(c)$ when $c \in (a, b]$, whence the assertion follows.

Theorem 3.3.7. Substitution Theorem *Let $J := [\alpha, \beta]$ and let $\varphi : J \to \mathbb{R}$ have a continuous derivative on J. If $f : I \to \mathbb{R}$ is continuous on an interval I containing $\varphi(J)$, then*

$$\int_{\alpha}^{\beta} f(\varphi(t)) \cdot \varphi'(t)dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx.$$

Proof. Let f and ϕ be two functions satisfying above hypothesis that f is continuous on I and $\phi^{'}$ is integrable on $[\alpha, \beta]$. Then the function $f(\phi(t))\phi^{'}(t)$ is also integrable on $[\alpha, \beta]$. Hence the integrals $\int_{\alpha}^{\beta} f(\phi(t))\phi^{'}(t)$ and $\int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx$ in fact exist. Since f is continuous on I, it has an antiderivative F by Theorem 3.3.5, where $F(x) = \int_{\phi(\alpha)}^{x} f(t)dt$. The composite function $F \circ \phi$ is then defined. Since ϕ is differentiable, combining the chain rule, Theorem 1.5.4 we get,

$$(F \circ \phi)^{'}(t) = F^{'}(\phi(t))\phi^{'}(t) = f(\phi(t))\phi^{'}(t).$$

Applying Fundamental Theorem 3.3.1 for $(F \circ \phi)^{'}(t) = f(\phi(t))\phi^{'}(t)$,

$$\int_{\alpha}^{\beta} f(\phi(t))\phi^{'}(t)dt = \int_{\alpha}^{\beta} (F \circ \phi)^{'}(t)dt$$

$$
\begin{aligned}
&= (F \circ \phi)(\beta) - (F \circ \phi)(\alpha) \\
&= F(\phi(\beta)) - F(\phi(\alpha)) \\
&= \int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx
\end{aligned}
$$

□

ample 3.3.8.

Consider the integral $\int_1^4 \frac{\sin\sqrt{t}}{\sqrt{t}} dt$.

Here we substitute $\varphi(t) := \sqrt{t}$ for $t \in [1,4]$ so that $\varphi'(t) = 1/(2\sqrt{t})$ is continuous on $[1,4]$. If we let $f(x) := 2\sin x$, then the integrand has the form $(f \circ \varphi) \cdot \varphi'$ and the Substitution Theorem 3.3.7 implies that the integral equals $\int_1^2 2\sin x dx = -\ 2\cos x]_1^2 = 2(\cos 1 - \cos 2)$.

Consider the integral $\int_0^4 \frac{\sin\sqrt{t}}{\sqrt{t}} dt$.

Since $\varphi(t) := \sqrt{t}$ does not have a continuous derivative on $[0,4]$, the Substitution Theorem 3.3.7 is not applicable, at least with this substitution. (In fact, it is not obvious that this integral exists; however, we can apply Exercise 2.2.11 to obtain this conclusion. We could then apply the Fundamental Theorem 3.3.1 to $F(t) := -2\cos\sqrt{t}$ with $E := \{0\}$ to evaluate this integral.)

Unit-III

Example Construct the forward difference table for the following x values and its corresponding f values.

x	0.1	0.3	0.5	0.7	0.9	1.1	1.3
f	0.003	0.067	0.148	0.248	0.370	0.518	0.697

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
0.1	0.003					
		0.064				
0.3	0.067		0.017			
		0.081		0.002		
0.5	0.148		0.019		0.001	
		0.100		0.003		0.000
0.7	0.248		0.022		0.001	
		0.122		0.004		0.000
0.9	0.370		0.026		0.001	
		0.148		0.005		
1.1	0.518		0.031			
		0.179				
1.3	0.697					

Example If Δ, ∇, δ denote forward, backward and central difference operators, E and μ respectively the shift operator and average operators, in the analysis of data with equal spacing h, prove the following:

(i) $1+\delta^2\mu^2 = \left(1+\frac{\delta^2}{2}\right)^2$ (ii) $E^{1/2} = \mu + \frac{\delta}{2}$

(iii) $\Delta = \frac{\delta^2}{2} + \delta\sqrt{1+(\delta^2/4)}$

(iv) $\mu\delta = \frac{\Delta E^{-1}}{2} + \frac{\Delta}{2}$ (v) $\mu\delta = \frac{\Delta+\nabla}{2}$.

Solution

(i) From the definition of operators, we have (*iv*) We write

$$\mu\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right) = \frac{1}{2}\left(E - E^{-1}\right)$$

$$= \frac{1}{2}\left(1 + \Delta - E^{-1}\right) = \frac{\Delta}{2} + \frac{1}{2}\left(1 - E^{-1}\right) = \frac{\Delta}{2} + \frac{1}{2}\left(\frac{E-1}{E}\right) = \frac{\Delta}{2} + \frac{\Delta}{2E}.$$

(*v*) We can write

$$\mu\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right) = \frac{1}{2}\left(E - E^{-1}\right)$$

$$= \frac{1}{2}\left(1 + \Delta - (1 - \nabla)\right) = \frac{1}{2}(\Delta + \nabla).$$

$$\mu\delta = \frac{1}{2}\left(E^{1/2} + E^{\ 1/2}\right)\left(E^{1/2} - E^{\ 1/2}\right) = \frac{1}{2}\left(E - E^{\ 1}\right).$$

Therefore

$$1 + \mu^2\delta^2 = 1 + \frac{1}{4}\left(E^2 - 2 + E^{-2}\right) = \frac{1}{4}\left(E + E^{-1}\right)^2$$

Also,

$$1 + \frac{\delta^2}{2} = 1 + \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)^2 = \frac{1}{2}\left(E + E^{-1}\right)$$

From equations (1) and (2), we get

$$1 + \delta^2\mu^2 = \left(1 + \frac{\delta^2}{2}\right)^2.$$

(*ii*) $\mu + \frac{\delta}{2} = \frac{1}{2}\left(E^{1/2} + E^{-1/2} + E^{1/2} - E^{-1/2}\right) = E^{1/2}.$

(*iii*) We can write

$$\frac{\delta^2}{2} + \delta\sqrt{1 + \left(\delta^2/4\right)} = \frac{\left(E^{1/2} - E^{-1/2}\right)^2}{2} + \left(E^{1/2} - E^{-1/2}\right)\sqrt{1 + \frac{1}{4}\left(E^{1/2} - E^{-1/2}\right)^2}$$

$$= \frac{E - 2 + E^{-1}}{2} + \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)\left(E^{1/2} + E^{-1/2}\right)$$

$$= \frac{E - 2 + E^{-1}}{2} + \frac{E - E^{-1}}{2}$$

$$= E \quad - 1$$

$$= \Delta$$

Example Prove that

$$hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta).$$

Using the standard relations given in boxes in the last section, we have

$$hD = \log E = \log(1+\Delta) = \log E = -\log E^{-1} = -\log(1+\nabla)$$

Also,

$$\mu\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right) = \frac{1}{2}\left(E + E^{-1}\right)$$

$$= \frac{1}{2}\left(e^{hD} - e^{-hD}\right) = \sin(hD)$$

Therefore

$$hD = \sinh^{-1}(\mu\delta).$$

Example Using the method of separation of symbols, show that

$$\Delta^n u_{x-n} = u_x - nu_{x-1} + \frac{n(n-1)}{2}u_{x-2} + \cdots + (-1)^n u_{x-n}.$$

To prove this result, we start with the right-hand side. Thus,

$$\begin{aligned}
\text{R.H.S} &= u_x - nu_{x-1} + \frac{n(n-1)}{2}u_{x-2} + \cdots + (-1)^n u_{x-n}. \\
&= u_x - nE^{-1}u_x + \frac{n(n-1)}{2}E^{-2}u_x + \cdots + (-1)^n E^{-n}u_x \\
&= \left[1 - nE^{-1} + \frac{n(n-1)}{2}E^{-2} + \cdots + (-1)^n E^{-n}\right]u_x \\
&= \left(1 - E^{-1}\right)^n u_x \\
&= \left(1 - \frac{1}{E}\right)^n u_x \\
&= \left(\frac{E-1}{E}\right)^n u_x \\
&= \frac{\Delta^n}{E^n}u_x \\
&= \Delta^n E^{-n}u_x \\
&= \Delta^n u_{x-n}, \\
&= \text{L.H.S}
\end{aligned}$$

Example Using Newton's forward difference interpolation formula and the following table evaluate $f(15)$.

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
10	46				
		20			
20	66		-5	2	
		15			
30	81		-3	-1	-3
		12			
40	93		-4		
		8			
50	101				

Here $x = 15$, $x_0 = 10$, $x_1 = 20$, $h = x_1 - x_0 = 20 - 10 = 10$, $r = (x - x_0)/h = (15–10)/10 = 0.5$, $f_0 = -46$, $\Delta f_0 = 20$, $\Delta^2 f_0 = -5$, $\Delta^3 f_0 = 2$, $\Delta^4 f_0 = -3$.

Substituting these values in the Newton's forward difference interpolation formula for $n = 4$, we obtain

$$f(x) \approx P_4(x) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \ldots + \frac{r(r-1)\ldots(r-4+1)}{4!}\Delta^4 f_0,$$

so that

$$f(15) \approx 46 + (0.5)(20) + \frac{(0.5)(0.5-1)}{2!}(-5) + \frac{(0.5)(0.5-1)(0.5-2)}{3!}(2)$$

$$+\frac{(0.5(0.5-1)(0.5-2)(0.5-3)}{4!}(-3)$$

= 56.8672, correct to 4 decimal places.

Example Values of x (in degrees) and $\sin x$ are given in the following table:

x (in degrees)	$\sin x$
15	0.2588190
20	0.3420201
25	0.4226183
30	0.5
35	0.5735764
40	0.6427876

Determine the value of $\sin 38^0$.

Solution

The difference table is

x	$\sin x$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
15	0.2588190					
		0.0832011				
20	0.3420201		−0.0026029			
		0.0805982		−0.0006136		
25	0.4226183		−0.0032165		0.0000248	
		0.0773817		−0.0005888		0.0000041
30	0.5		−0.0038053		0.0000289	
		0.0735764		−0.0005599		
35	0.5735764					
			−0.0043652			
		0.0692112				
40	0.6427876					

As 38 is closer to $x_n = 40$ than $x_0 = 15$, we use Newton's backward difference formula with

$$r = \frac{x - x_n}{h} = \frac{38 - 40}{5} = -\frac{2}{5} = -0.4$$

Hence, using formula, we obtain

$$f(38) = 0.6427876 - 0.4(0.0692112) + \frac{-0.4(-0.4-1)}{2}(-0.0043652)$$

$$+\frac{(-0.4)(-0.4+1)(-0.4+2)}{6}(-0.0005599)$$

$$+\frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)}{24}(0.0000289)$$

$$+\frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)(-0.4+4)}{120}(0.0000041)$$

$$= 0.6427876 - 0.02768448 + 0.00052382 + 0.00003583$$

$$-0.00000120$$

$$= 0.6156614$$

Example Using the following table find f (x) as a polynomial in *x*

x	f (x)
–1	3
0	– 6
3	39
6	822
7	1611

The divided difference table is

x	f (x)	$f[x_k, x_{k+1}]$			
–1	3				
		–9			
0	–6		6		
		15		5	
3	39		41		1
		261		13	
6	822		132		
		789			
7	1611				

Hence

$$f(x) = 3 + (x+1)(-9) + x(x+1)(6) + x(x+1)(x-3)(5)$$
$$+x(x+1)(x-3)(x-6)$$
$$= x^4 - 3x^3 + 5x^2 - 6.$$

Example Find the interpolating polynomial by Newton's divided difference formula for the following table and then calculate *f*(2.1).

x	0	1	2	4
f(*x*)	1	1	2	5

.

x	*f*(*x*)	First divided difference $f[x_{k-1}, x_k]$	Second divided difference $f[x_{k-1}, x_k, x_{k+1}]$	Third divided difference $f[x_{k-1}, x_k, x_{k+1}, x_{k+2}]$
0	1	$f(x_0, x_1) = 0$		
1	1	$f(x_1, x_2) = 1$	$-1/2$	
2	2	$f(x_2, x_3) = 3/2$	$-1/6$	$-\frac{1}{2}$
4	5			

Now substituting the values in the formula, we get

$$f(x) \approx 1 + (x-0)(0) + (x-0)(x-1)\left(\frac{1}{2}\right) + (x-0)(x-1)(x-2)\left(-\frac{1}{12}\right)$$

$$= -\frac{1}{12}x^3 + \frac{3}{4}x^2 - \frac{2}{3}x + 1$$

Substituting $x = 2.1$ in the above polynomial, we get $f(2.1)$=2.135,

Problem 3: Obtain Newton's divided difference interpolating polynomial satisfying the following values:

x:	1	3	4	5	7	10
f(x):	3	31	69	131	351	1011

Also find f(4.5), f(8) and the second derivative of f(x) at x=3.2.

Solution:

To obtain the Newton's divided difference interpolating polynomial f(x), we need the divided difference using the given values.

It is calculated and listed in the following table

X	First divided differences	Second divided differences	Third divided differences	Fourth divided differences
1				
	14			
3		**8**		
	38		**1**	
4		12		0
	62		1	
5		16		0
	110		1	
7		22		
	220			
9				

Since the fourth divided differences are zeroes, f(x) is of degree 3 and it is obtained as,

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)$$
$$+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)$$

$$f(x_0)=f(1)=3;\ f(x_0,x_1)=14;\ f(x_0,x_1,x_2)=8 \quad and \quad f(x_0,x_1,x_2,x_3)=1$$

$$\Rightarrow \ f(x)=3+(x-1)\times 14+(x-1)(x-3)\times 8+(x-1)(x-3)(x-4)\times 1$$

That is,

$$f(x)=x^3+x+1$$

Hence, $f(4.5)=(4.5)^3+4.5+1=96.625$ and $f(8)=(8)^3+8+1=521$

Second derivative of $f(x)$ *is* $6x$. Now second derivative of f(x) at x=3.2 is $6\times 3.2=19.2$

Problem : Given f(2) = 9, and f(6) = 17. Find an approximate value for f(5) by the method of Lagrange's interpolation.

Solution:

For the given two points (2,9) and (6,17), the Lagrangian polynomial of degree 1 is $p_1(x) = l_0(x) f(x_0) + l_1(x) f(x_1)$, where, $l_0(x) = \frac{(x - x_1)}{(x_0 - x_1)}$ and $l_1(x) = \frac{(x - x_0)}{(x_1 - x_0)}$. That is,

$$p_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1)$$

$$\Rightarrow \ p_1(x) = \frac{(x-6)}{(2-6)} \times 9 + \frac{(x-2)}{(6-2)} \times 17$$

Hence,

$$f(5) = \ P_1(5) = \frac{(5-6)}{(2-6)} \times 9 + \frac{(5-2)}{(6-2)} \times 17$$

$$= \frac{1}{4} \times 9 + \frac{3}{4} \times 17$$

$$= 15$$

Example If $y_1 = 4,\ y_3 = 12,\ y_4 = 19$ and $y_x = 7,$ find x. Compare with the actual value.

Using the inverse interpolation formula,

$$x \approx L_n(7) = \sum_{k=0}^{2} \frac{l_k(7)}{l_k(y_k)} x_k$$

where $x_0 = 1,\ y_0 = k,\quad x_y = 3,\ y_1 = 12\quad x_2 = 4,\ y_2 = 19$ and $y = 7$

i.e., $x \approx \frac{(7 - y_1)\,(7 - y_2)}{(y_0 - y_1)\,(y_0 - y_2)} x_0 + \frac{(7 - y_0)\,(7 - y_2)}{(y_1 - y_0)\,(y_1 - y_2)}$

$$x_1 + \frac{(7 - y_0)\,(7 - y_1)}{(y_2 - y_0)\,(y_2 - y_1)} x_2$$

i.e., $x = \frac{(-5)\,(-12)}{(-8)\,(-15)}(1) + \frac{(3)\,(-12)}{(8)\,(-7)}(3) + \frac{(3)\,(-5)}{(15)\,(7)}(4)$

$$= \frac{1}{2} + \frac{27}{14} - \frac{4}{7}$$

$$= 1.86$$

The actual value is 2.0 since the above values were obtained from the polynomial $y(x) = x^2 + 3$.

Example Compute $f'(0.2)$ and $f''(0)$ from the following tabular data.

x	0.0	0.2	0.4	0.6	0.8	1.0
$f(x)$	1.00	1.16	3.56	13.96	41.96	101.00

Since $x = 0$ and 0.2 appear at and near beginning of the table, it is appropriate to use formulae based on forward differences to find the derivatives. The forward difference table for the given data is:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0.0	1.00					
		0.16				
0.2	1.16		2.24			
		2.40		5.76		
0.4	3.56		8.00		3.84	
		10.40		9.60		0.00
0.6	13.96		17.60		3.84	
		28.00		13.44		
0.8	41.96		31.04			
		59.04				
1.0	101.00					

Using $$f'(x) = Df(x) = \frac{1}{h}\left(\Delta f(x) - \frac{\Delta^2 f(x)}{2} + \frac{\Delta^3 f(x)}{3} - \frac{\Delta^4 f(x)}{4} + \ . \ . \ . \ \right)$$

we obtain

$$f'(0.2) = \frac{1}{0.2}\left[2.40 - \frac{8.00}{2} + \frac{9.60}{3} - \frac{3.84}{4}\right] = 3.2$$

Using

$$f''(x) = D^2 f(x) = \frac{1}{h^2}\left(\Delta^2 f(x) - \Delta^3 f(x) + \frac{11}{12}\Delta^4 f(x) - \ . \ . \ . \ \right)$$

we obtain

$$f''(0) = \frac{1}{(0.2)^2}\left[2.24 - 5.76 + \frac{11}{12}(3.84) - \frac{5}{6}(0)\right] = 0.0$$

Example From the following table of values of x and y, obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $x = 1.2$:

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

The difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	2.7183						
		0.6018					
1.2	3.3201		0.133				
		0.7351		0.0294			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	0.001
1.6	4.9530		0.1988		0.0080		
		1.0966		0.0441		0.0014	
1.8	6.0496		0.2429		0.0094		
		1.3395		0.0535			
2.0	7.3891		0.2964				
		1.6359					
2.2	9.0250						

Here $x = 1.2$, $f(x) = 3.3201$ and $h = 0.2$. Hence

$$\left[\frac{dy}{dx}\right]_{x=1.2} = f'(1,2)$$

$$= \frac{1}{0.2}\left[0.7351 - \frac{1}{2}(0.1627) + \frac{1}{3}(0.0361) - \frac{1}{4}(0.0080) + \frac{1}{5}(0.0014)\right]$$

$$= 3.3205.$$

Similarly, $\left[\frac{d^2y}{dx^2}\right]_{x=1.2} = \frac{1}{0.04}\left[0.1627 - 0.0361 + \frac{11}{12}(0.0080) - \frac{5}{6}(0.0014)\right] = 3.318.$

For example: We find the first derivative of a function at 0, using the points $(-4,1245),(-1,33),(0,5),(2,9)$ *and* $(5,1335)$ where x values are not equidistant. We can get the approximating polynomial by Newton's divided difference formula.

The table of divided differences is,

x	y	First divided differences	Second divided differences	Third divided differences	Fourth divided differences
-4	1245				
		-404			
-1	33				
		-28	94		
0	5				
		2		-14	
2	9		10		3
		442		13	
5	1335				
			88		

Given $f(x_0)=1245$. From the table, we can observe that

$$f(x_0,x_1)=-404;\quad f(x_0,x_1,x_2)=94;$$
$$f(x_0,x_1,x_2,x_3)=-14 \quad and \quad f(x_0,x_1,x_2,x_3,x_4)=3$$

Hence the interpolating polynomial is

$$f(x)=1245+(x-(-4))\times(-404)+(x-(-4))(x-(-1))\times 94$$
$$+(x-(-4))(x-(-1))(x-0)\times(-14)+(x-(-4))(x-(-1))(x-0)(x-2)\times 3$$

On simplification, we get

$$f(x)=3x^4-5x^3+6x^2-14x+5\,.$$

Then,

$$f'(x)=12x^3-15x^2+12x-14$$

Hence, $$f'(0)=-14\,.$$

UNIT-IV

Example Use the trapezoidal rule with $n=4$ to estimate

$$\int_1^2 \frac{1}{x}\,dx.$$

Compare the estimate with the exact value of the integral.

To find the trapezoidal approximation, we divide the interval of integration into four subintervals of equal length and list the values (correct to five decimal places) of $y=\frac{1}{x}$ at the endpoints and partition points.

j	x_j	$y_j=\frac{1}{x_j}$	
0	1.0	1.00000	
1	1.25		0.80000
2	1.50		0.66667
3	1.75		0.57143
4	2.00	0.50000	
	Sum	1.50000	2.0381

With $n=4$ and $h=\frac{b-a}{n}=\frac{2-1}{4}=\frac{1}{4}=0.25$:

$$T=\frac{h}{2}[y_0+y_4+2(y_1+y_2+y_3)]$$

$$=\frac{1}{8}\left[1.5+2(2.0381)\right]=0.69702.$$

The exact value of the integral is

$$\int_1^2 \frac{1}{x}dx=\ln x\Big]_1^2=\ln 2-\ln 1=0.69315$$

The approximation is a slight overestimate.

Example Evaluate $\int_0^1 e^{-x^2}\,dx$ by means of Trapezoidal rule with n=10.

Here $h = \frac{b-a}{n} = \frac{1-0}{1} = 0.1$ and

$$\int_0^1 e^{-x^2}\,dx \approx T = \frac{0.1}{2}\left[y_0 + y_{10} + 2(y_1 + y_2 + \cdots + y_9)\right]$$

j	x_j	x_j^2	$f(x_j) = e^{-x_j^2}$	
0	0.0	0.00	1.000 000	0.990 050
1	0.1	0.01		0.960 789
2	0.2	0.04		0.913 931
3	0.3	0.09		0.852 144
4	0.4	0.16		0.778 801
5	0.5	0.25		0697 676
6	0.6	0.36		0.612 626
7	0.7	0.49		0.612 626
8	0.8	0.64		0.527 292
9	0.9	0.81		0.444 858
10	1.0	1.00	0.367 879	
	Sums		1.367 879	6.778 167

Hence $\int_0^1 e^{-x^2}\,dx \approx T = \frac{0.1}{2}\left[1.367879 + 2(6.778167)\right] = 0.746211$

Example Find an approximate value of $\log_e 5$ by calculating $\int_0^5 \frac{dx}{4x+5}$, by Simpson's 1/3 rule of integration.

We note that

$$\int_0^5 \frac{dx}{4x+5} = \left[\frac{1}{4}\log(4x+5)\right]_0^5 = \frac{1}{4}[\log 25 - \log 5] = \frac{1}{4}\log\frac{25}{5} = \frac{1}{4}\log 5.$$

Now to calculate the value of $\int_0^5 \frac{dx}{4x+5}$, by Simpson's rule of integration, divide the interval [0, 5] into n = 10 equal subintervals, each of length $h = \frac{b-a}{n} = \frac{5-0}{10} = 0.5.$

j	x_j	$4x_j+5$	$f_j = f(x_j) = \frac{1}{4x_j+5}$		
0	0.0	5	0.20		
1	0.5	7		0.1429	
2	1.0	9			0.1111
3	1.5	11		0.0909	
4	2.0	13			0.0769
5	2.5	15		0.6666	
6	3.0	17			0.0588
7	3.5	19		0.0526	
8	4.0	21			0.0476
9	4.5	23		0.0434	
10	5.0	25	0.04		
Sums			s_0=0.24	s_1=0.3963	s_2=0.2944

Hence,

$$\int_0^5 \frac{dx}{4x+5} \approx S = \frac{0.5}{3}[0.24 + 4(0.3963) + 2(0.2944)] = 0.4023.$$

and $\log_e 5 = 4(0.4023) = 1.6092.$

Problem: Find $\int_0^{10} \frac{1}{1+x^2}dx$ using Simpson's one third rule.

Solution:

By Simpson's one third rule, $\int_a^b f(x)dx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + ...) + 2(y_2 + y_4 + ...) + y_n\right]$

In our integral, $\int_0^{10} \frac{1}{1+x^2}dx$, let the range [0,10] is subdivided into 10 equal interval of width *h=1*, by the x values 0,1,2,3,4,5,6,7,8,9 and 10. Corresponding y values of the function $\frac{1}{1+x^2}$ are listed below:

x	0	1	2	3	4	5	6	7	8	9	10
y	1	0.5	0.2	0.1	0.0588	0.0385	0.0270	0.02	0.0154	0.0122	0.0099

$$\int_0^{10} \frac{1}{1+x^2} dx = \frac{1}{3}\left[1+4(0.5+0.1+0.0385+0.02+0.0122)+2(0.2+0.0588+0.027+0.0154)+0.0099\right]$$

$$= \frac{1}{3}\left[1.0099+4(0.6707)+2(0.3012)\right]$$

$$= \frac{1}{3}[4.2951] = 1.4317.$$

Problem: Evaluate $\int_0^6 \frac{1}{3+x^2} dx$ using Simpson's three eight rule.

Solution:

By Simpson's three eight rule,

$$\int_a^b f(x)dx = \frac{3h}{8}\left[y_0+3(y_1+y_2+y_4+....+y_{n-1})+2(y_3+y_6+y_9+...)+y_n\right]$$

Let the limit of integral [0,6] be divided into six equal parts with interval *h=1*, using the x values 0,1,2,3,4,5 and 6. Corresponding y values of the given integrand function $\frac{1}{3+x^2}$ are,

x	0	1	2	3	4	5	6
y	0.333	0.25	0.1429	0.1	0.0526	0.0357	0.0256

Thus,

$$\int_0^6 \frac{1}{3+x^2} dx = \frac{3\times 1}{8}\left[y_0+3(y_1+y_2+y_4+....+y_{n-1})+2(y_3+y_6+y_9+...)+y_n\right]$$

For n=6,

$$\int_0^6 \frac{1}{3+x^2} dx = \frac{3\times 1}{8}\left[y_0+3(y_1+y_2+y_4+y_5)+2y_3+y_6\right]$$

For n=6,

$$\int_0^6 \frac{1}{3+x^2} dx = \frac{3\times 1}{8}\left[y_0+3(y_1+y_2+y_4+y_5)+2y_3+y_6\right]$$

$$\int_0^6 \frac{1}{3+x^2} dx = \frac{3\times 1}{8}\left[0.333+3(0.25+0.1429+0.0526+0.0357)+2(0.1)+0.0256\right]$$

$$= \frac{3}{8}[0.333+1.4436+0.2+0.0256] = \frac{3}{8}[2.0022]$$

$$\Rightarrow \int_0^6 \frac{1}{3+x^2} dx = 0.7508.$$

Fig.4

Example Compute the integral $I = \sqrt{\frac{2}{\pi}} \int_0^1 e^{-x^2/2} dx$ using Simpson's 1/3 rule, taking $h = 0.125$.

j	x_j	$f_j = f(x_j) = \sqrt{\frac{2}{\pi}} e^{-x_j^2/2}$		
0	0.000	0.7979		
1	0.125		0.7917	
2	0.250			0.7733
3	0.375		0.7437	
4	0.500			0.7041
5	0.625		0.6563	
6	0.750			0.6023
7	0.875		0.5441	
8	1.000	0.4839		
Sums		s_0=1.2818	s_1=2.7358	s_2=2.0797

Hence $I = \sqrt{\frac{2}{\pi}} \int_0^1 e^{-x^2/2} dx \approx S = \frac{0.125}{3}[1.2818 + 4(2.7358) + 2(2.0797)]$

$$= 0.6827$$

Example Find a real root of the equation $x^3 + x^2 - 1 = 0$ on the interval $[0, 1]$ with an accuracy of 10^{-4}.

To find this root, we rewrite the given equation in the form

$$x = \frac{1}{\sqrt{x+1}}$$

Take

$$\phi(x) = \frac{1}{\sqrt{x+1}}. \text{ Then } \phi(x) = -\frac{1}{2}\frac{1}{(x+1)^{\frac{3}{2}}}$$

$$\max_{[0,1]} |\phi'(x)| = |\frac{1}{2\sqrt{8}}| = k = 0.17678 < 0.2.$$

Choose $\phi(x) = 3 - \frac{1}{x}$. Then $\phi'(x) = \frac{1}{x^2}$ and $|\phi'(x) <| 1$ on the interval $(1, 2)$.

Hence the iteration method gives:

n	x_n	$\sqrt{x_n+1}$	$x_{n+1} = 1/\sqrt{x_n+1}$
0	0.75	1.3228756	0.7559289
1	0.7559289	1.3251146	0.7546517
2	0.7546617	1.3246326	0.7549263

At this stage,

$$|x_{n+1} - x_n| = 0.7549263 - 0.7546517 = 0.0002746,$$

which is less than 0.0004. The iteration is therefore terminated and the root to the required accuracy is 0.7549.

Example Find the root of the equation $2x = \cos x + 3$ correct to three decimal places.

We rewrite the equation in the form

$$x = \frac{1}{2}(\cos x + 3)$$

so that

$$\phi = \frac{1}{2}(\cos x + 3),$$

and

$$|\phi'(x)| = \left|\frac{\sin x}{2}\right| < 1.$$

Hence the iteration method can be applied to the eq. (3) and we start with $x_0 = \pi/2$. The successive iterates are

$$x_1 = 1.5, \quad x_2 = 1.535, \quad x_3 = 1.518,$$
$$x_4 = 1.526, \quad x_5 = 1.522, \quad x_6 = 1.524,$$
$$x_7 = 1.523, \quad x_8 = 1.524.$$

We accept the solution as 1.524 correct to three decimal places.

Example Solve the equation $x^3 = \sin x$. Considering various $\phi(x)$, discuss the convergence of the solution.

How do the functions we considered for $\phi(x)$ compare? Table shows the results of several

iterations using initial value $x_0 = 1$ and four different functions for $\phi(x)$. Here x_n is the value of x

on the nth iteration .

Answer:

When $\phi(x) = \sqrt[3]{\sin x}$, we have:

$x_1 = 0.94408924124306$; $x_2 = 0.93215560685805$

$x_3 = 0.92944074461587$; $x_4 = 0.92881472066057$

When $\phi(x) = \frac{\sin x}{x^2}$, we have:

$x_1 = 0.84147098480790$; $x_2 = 1.05303224555943$

$x_3 = 0.78361086350974$; $x_4 = 1.14949345383611$

Referring to Theorem, we can say that for $\phi(x) = \frac{\sin x}{x^2}$, the iteration doesn't converge.

When $\phi(x) = x + \sin x - x^3$, we have:

$x_1 =$ 0.84147098480790; $x_2 = 0.99127188988250$

$x_3 = 0.85395152069647$; $x_4 = 0.98510419085185$

When $\phi(x) = x - \frac{\sin x - x^3}{\cos x - 3x^2}$, we have:

$x_1 =$ 0.93554939065467; $x_2 = 0.92989141894368$

$x_3 = 0.92886679103170$; $x_4 = 0.92867234089417$

Example Find a real root of the equation $f(x) = x^3 - x - 1 = 0$.

Since $f(1)$ is negative and $f(2)$ positive, a root lies between 1 and 2 and therefore we take $x_0 = 3/2 = 1.5$. Then

$f(x_0) = \frac{27}{8} - \frac{3}{2} = \frac{15}{8}$ is positive and hence $f(1)\, f(1.5) < 0$ and Hence the root lies between 1 and 1.5 and we obtain

$$x_1 = \frac{1 + 1.5}{2} = 1.25$$

$f(x_1) = -19/64$, which is negative and hence $f(1)\, f(1.25) > 0$ and hence a root lies between 1.25 and 1.5. Also,

$$x_2 = \frac{1.25 + 1.5}{2} = 1.375$$

The procedure is repeated and the successive approximations are

$x_3 = 1.3125$, $x_4 = 1.34375$, $x_5 = 1.328125$, etc.

Example Find a positive root of the equation $xe^x = 1$, which lies between 0 and 1.

Let $f(x) = xe^x - 1$. Since $f(0) = -1$ and $f(1) = 1.718$, it follows that a root lies between 0 and 1. Thus,

$$x_0 = \frac{0+1}{2} = 0.5.$$

Since $f(0.5)$ is negative, it follows that a root lies between 0.5 and 1. Hence the new root is 0.75, i.e.,

$$x_1 = \frac{.5+1}{2} = 0.75.$$

Since $f(x_1)$ is positive, a root lies between 0.5 and 0.75. Hence

$$x_2 = \frac{.5+.75}{2} = 0.625$$

Since $f(x_2)$ is positive, a root lies between 0.5 and 0.625. Hence

$$x_3 = \frac{.5+.625}{2} = 0.5625.$$

We accept 0.5625 as an approximate root.

Example Given that the equation $x^{2.2} = 69$ has a root between 5 and 8. Use the method of regula-falsi to determine it.

Let $f(x) = x^{2.2} - 69$. We find

$$f(5) = -3450675846 \text{ and } f(8) = -28.00586026.$$

$$x_1 = \frac{\begin{vmatrix} 5 & 8 \\ f(5) & f(8) \end{vmatrix}}{f(8) - f(5)} = \frac{5(28.00586026) - 8(-34.50675846)}{28.00586026 + 34.50675846} = 6.655990062.$$

Now, $f(x_1) = -4.275625415$ and therefore, $f(5)\, f(x_1) > 0$ and hence the root lies between 6.655990062 and 8.0. Proceeding similarly,

$$x_2 = 6.83400179, \quad x_3 = 6.850669653,$$

The correct root is $x_3 = 6.8523651\cdots$, so that x_3 is correct to these significant figures. We accept 6.850669653 as an approximate root.

Example Apply Newton's method to solve the algebraic equation $f(x) = x^3 + x - 1 = 0$ correct to 6 decimal places. (Start with x_0=1)

$$f(x) = x^3 + x - 1,$$

$$f'(x) = 3x^2 + 1$$

and substituting these in Newton's iterative formula, we have

$$x_{n+1} = x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1} \quad \text{or} \quad x_{n+1} = \frac{2x_n^3 + 1}{3x_n^2 + 1}, \; n = 0,1,2,\ldots.$$

Starting from x_0=1.000 000,

$x_1 + 0.750000,\ x_2 = 0.686047,\quad x_3 = 0.682340,\ x_4 = 0.682328, \cdots$ and we accept 0.682328 as an approximate solution of $f(x) = x^3 + x - 1 = 0$ correct to 6 decimal places.

Example Set up Newton-Raphson iterative formula for the equation

$$x \log_{10} x - 1.2 = 0.$$

Solution

Take $\qquad f(x) = x \log_{10} x - 1.2.$

Noting that $\log_{10} x = \log_e x \cdot \log_{10} e \approx 0.4343 \log_e x,$

we obtain $\qquad f(x) = 0.4343 x \log_e x - 1.2.$

$$f'(x) = 0.4343 \log_e x + 0.4343 x \times \frac{1}{x} = \log_{10} x + 0.4343$$

and hence the Newton's iterative formula for the given equation is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{0.4343 x \log_e x_n - 1.2}{\log_{10} x + 0.4343}.$$

Example Find the positive solution of the transcendental equation

$$2\sin x = x.$$

Here $\quad f(x) = x - 2\sin x,$

so that $\quad f'(x) = 1 - 2\cos x$

Substituting in Newton's iterative formula, we have

$$x_{n+1} = x_n - \frac{x_n - 2\sin x_n}{1 - 2\cos x_n}, \quad n = 0, 1, 2, \cdots \quad \text{or}$$

$$x_{n+1} = \frac{2(\sin x_n - x_n \cos x_n)}{1 - 2\cos x_n} = \frac{N_n}{D_n}, \quad n = 0, 1, 2, \cdots$$

where we take $N_n = 2(\sin x_n - x_n \cos x_n)$ and $D_n = 1 - 2\cos x_n$, to easy our calculation. Values calculated at each step are indicated in the following table (Starting with $x_0 = 2$).

n	x_n	N_n	D_n	x_{n+1}
0	2.000	3.483	1.832	1.901
1	1.901	3.125	1.648	1.896
2	1.896	3.107	1.639	1.896

1.896 is an approximate solution to $2\sin x = x$.

Example Find a real root of the equation $x = e^{-x}$, using the Newton – Raphson method.

$$f(x) = xe^x - 1 = 0$$

Let $x_0 = 1$. Then

$$x_1 = 1 - \frac{e-1}{2e} = \frac{1}{2}\left(1 + \frac{1}{e}\right) = 0.6839397$$

Now $\quad f(x_1) = 0.3553424$, and $f'(x_1) = 3.337012$,

$$x_2 = 0.6839397 - \frac{0.3553424}{3.337012} = 0.5774545.$$

$$x_3 = 0.5672297 \text{ and } x_4 = 0.5671433.$$

Example Find a real root of the equation $x^3 - 2x - 5 = 0$ using secant method.

Let the two initial approximations be given by $x_{-1} = 2$ and $x_0 = 3$.

We have

$$f(x_{-1}) = f_1 = 8 - 9 = -1, \quad \text{and} \quad f(x_0) = f_0 = 27 - 11 = 16.$$

$$x_1 = \frac{2(16) - 3(-1)}{17} = \frac{35}{17} = 2.058823529.$$

Also,

$$f(x_1) = f_1 = -0.390799923.$$

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{3(-0.390799923) - 2.058823529(16)}{-16.390799923} = 2.08126366.$$

Again

$$f(x_2) = f_2 = -0.147204057.$$

$$x_3 = 2.094824145.$$