Biyani Girl's College

Concept Based Notes

BSc Semester-III

Paper: Real analysis –I and Differential equation-i

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Preface

I am glad to present this book, especially designed to serve the needs of the students. The book has been written keeping in mind the general weakness in understanding the fundamental concepts of the topics. The book is self-explanatory and adopts the "Teach Yourself" style. It is based on question-answer pattern. The language of book is quite easy and understandable based on scientific approach.

Any further improvement in the contents of the book by making corrections, omission and inclusion is keen to be achieved based on suggestions from the readers for which the author shall be obliged.

I acknowledge special thanks to Mr. Rajeev Biyani, Chairman& Dr. Sanjay Biyani, Director (Acad.) Biyani Group of Colleges, who are the backbones and main concept provider and also have been constant source of motivation throughout this endeavor. They played an active role in coordinating the various stages of this endeavor and spearheaded the publishing work.

I look forward to receiving valuable suggestions from professors of various educational institutions, other faculty members and students for improvement of the quality of the book. The reader may feel free to send in their comments and suggestions to the under mentioned address.

Author

Detailed Syllabus

[UG0803-MAT-63T-201] - [Real Analysis-I & Differential Equations-I]

Unit - I

Bounded set, Neighbourhood, Limit point, Bolzano-Weierstrass theorem, closed and Open sets. Concept of compactness and connectedness. Heine-Borel theorem. (15 Lectures)

Unit - II

Real sequences- Limit and Convergence of a sequence, Monotonic sequences. Cauchy's sequences, Subsequences, Cauchy's general principle of convergence. Continuous functions: Properties of continuous functions on closed intervals. (15 Lectures)

Unit -III

Exact differential equations and equations which can be made exact. First order but higher degree differential equations solvable for x,y and p. Linear differential equations with constant coefficients, Complementary function and Particular integral. (15 Lectures)

Unit-IV

Homogeneous linear differential equations, Linear differential equations of second order. Solution by transformation of the equation by changing the dependent variable/the independent variable, Method of variation of parameters, Method of undetermined coefficients.
(15 Lectures)

Suggested Books and References –

1. Royden H, Fitzpatrick PM. Real analysis. China Machine Press; 2010.

2. Rudin W. Principles of mathematical analysis. New York: McGraw-hill; 1964. 3. Bartle RG, Sherbert DR. Introduction to real analysis. New York: Wiley; 2000. 4. Mapa SK. Introduction to Real Analysis. Sarat Book Distributors; 2014.

3. Raisinghania MD, Ordinary and partial differential equations. S. Chand Publishing; 2013.

UNIT-1

BOUNDED SETS

Definition : A subset A of R is said to be Bounded above if there exists an element α ϵ R such that Then α is called an upper bound of A. A is said to be bounded below if there exists an element β ϵ R such that . β is called a lower bound of A. A is said to be bounded if it is both bounded above and bounded below.

Least Upper Bound and Greatest Lower Bound: Definition: Let A be a subset of R and u is called the least upper bound or supremum of A if i. u is an upper bound of A. ii. if then v is not an upper bound of A. Let A is a subset of R and is called the greatest lower bound or infimum of A if i. l is a lower bound of A. ii. if then m is not a lower bound of A.

Examples: 1. Let A = {1, 3, 5, 6}. Then glb of A = 1 and lub of A = 6 2. Let A = (0,1). Then glb of A = 0 and lub of A = 1. In this case both glb and lub do not belong to A.

Bounded Functions: **Definition:** Let be any function. Then the range of f is a subset of R. f is said to be bounded function if its range is a bounded subset of R. Hence f is a bounded function iff there exists a real number m such that

Examples: 1. f : [0,1] R given by f(x) = x + 2 is a bounded function where as f : R given by is not a bounded function.

2. f :R R defined by f(x) =sin x is a bounded function. Since Absolute Value: Definition: For any real number x we defined the modulus or the absolute value of x denoted by |x| as follows .

Question 10. *How do limits of functions relate to limits of sequences?*

Theorem 11

Let $S \subset \mathbb{R}$, c a cluster point of S, and let $f : S \to \mathbb{R}$. Then, the following are equivalent:

1. $\lim_{x \to c} f(x) = L$ and
2. for every sequence $\{x_n\}$ in $S \setminus \{c\}$ such that $x_n \to c$, we have $f(x_n) = L$.

Proof: (1. $\implies$ 2.): Suppose $\lim_{x \to c} f(x) = L$. Let $\{x_n\}$ be a sequence in $S \setminus \{c\}$ such that $x_n \to c$. We want to show that $f(x_n) \to L$. Let $\epsilon > 0$. Given $\lim_{x \to c} f(x_n) = L$, $\exists \delta > 0$ such that if $x \in S$ and $0 < |x - c| < \delta$ then $|f(x) - L| < \epsilon$. Since $x_n \to c$, $\exists M_0 \in \mathbb{N}$ such that $\forall n \geq M_0$, $0 < |x_n - c| < \delta$.

Choose $M = M_0$. Then, $\forall n \geq M$, if $0 < |x_n - c| < \delta$ then $|f(x_n) - L| < \epsilon$. Thus, $f(x_n) \to L$.

(2. $\impliedby$ 1.): Suppose 2. holds, and assume for the sake of contradiction that 1) is false. Then, $\exists \epsilon_0 > 0$ such that $\forall \delta > 0$, $\exists x \in S$ such that

$$0 < |x - c| < \delta \ \text{ and } \ |f(x) - L| \geq \epsilon_0.$$

Then, $\forall n \in \mathbb{N}$, $\exists x_n \in S$ such that $0 < |x_n - c| < \frac{1}{n}$ and $|f(x_n) - L| \geq \epsilon_0$. By the Squeeze Theorem applied to

$$0 < |x_n - c| < \frac{1}{n},$$

$x_n \to c$. Then, by 2.,

$$0 = \lim_{n \to \infty} |f(x_n) - L| \geq \epsilon_0$$

which is a contradiction. □

Definition 1 (Bounded Functions)
A function $f : S \to \mathbb{R}$ is bounded if $\exists B \geq 0$ such that for all $x \in S$,

$$|f(x)| \leq B.$$

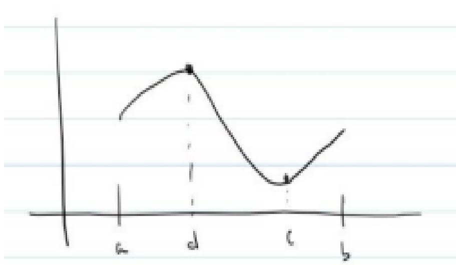
Theorem 2
If $f : [a, b] \to \mathbb{R}$ is continuous then f is bounded.

Proof: Suppose for the sake of contradiction that $f : [a, b] \to \mathbb{R}$ is continuous and f is unbounded. Then, $\forall n \in \mathbb{N}$, $\exists x_n \in [a, b]$ such that $|f(x_n)| \geq n$. By the Bolzano-Weierstrass theorem, $\exists$ a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ and an $x \in \mathbb{R}$ such that $x_{n_k} \to x$. Since $a \leq x_{n_k} \leq b$ for all k, $a \leq x \leq b$. Given f is continuous at x by assumption,

$$f(x) = \lim_{k\to\infty} f(x_{n_k}) \implies |f(x)| = \lim_{k\to\infty} |f(x_{n_k})|.$$

Therefore, $\{|f(x_{n_k})|\}$ is bounded, and thus $\{n_k\}$ is bounded since $n_k \leq |f(x_{n_k})|$. But by the definition of a subsequence, we must have $k \leq n_k$ for all k, contradicting the boundedness of $\{n_k\}$. □

Definition 3 (Absolute Minimum/Maximum)
Let $f : S \to \mathbb{R}$. Then, f achieves an absolute minimum at c if $\forall x \in S$, $f(x) \geq f(c)$. Similarly, f achieves an absolute maximum at d if $\forall x \in S$, $f(x) \leq f(d)$.



Bolzano-Weierstrass Theorem :

Statement: Every bounded infinite set of real numbers has a limit point.

Definition

A limit point of a set S is a point x such that every neighborhood of x contains infinitely many points of S.

Proof:

Let S be a bounded infinite set of real numbers. Then, S is contained in some closed interval [a, b].

We will construct a limit point using the following steps:

1. Divide the interval [a, b] into two equal subintervals: [a, (a+b)/2] and [(a+b)/2, b].

2. Since S is infinite, at least one of these subintervals contains infinitely many points of S. Choose one such subinterval and label it $[a_1, b_1]$.

3. Repeat step 1 with the subinterval $[a_1, b_1]$, dividing it into two equal subintervals.

4. Again, choose the subinterval that contains infinitely many points of S and label it $[a_2, b_2]$.

5. Continue this process, generating a sequence of nested intervals $[a_k, b_k]$ such that each interval contains infinitely many points of S.

Existence of a Limit Point

The sequence of nested intervals $[a_k, b_k]$ has a unique limit point c, which can be shown to be a limit point of S.

For any $\varepsilon > 0$, the interval $(c - \varepsilon, c + \varepsilon)$ contains infinitely many points of S, since it contains $[a_k, b_k]$ for sufficiently large k. Therefore, c is a limit point of S.

The result shows that every bounded infinite set of real numbers has a limit point, which is a fundamental property in real analysis.

Open Sets

A set S is open if for every point x in S, there exists a neighborhood around x that is entirely contained in S.

1. A set S is open if for every $x \in S$, $\exists\ \varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq S$.

2. Open sets can be thought of as sets where every point has some "wiggle room" around it.

Closed Sets

A set S is closed if it contains all its limit points.

1. A set S is closed if it contains all its limit points.

2. A limit point of S is a point x such that every neighborhood of x contains at least one point of S other than x itself.

Examples

Open Sets

1. The interval (0, 1) is open because for any $x \in (0, 1)$, you can find a small enough $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon)$ is contained in (0, 1).

2. The set of all real numbers, $\mathbb{R}$, is both open and closed.

Closed Sets

1. The interval [0, 1] is closed because it contains all its limit points, including 0 and 1.

2. The set {0, 1, 2} is closed because it contains all its limit points (in this case, none outside the set itself).

Important Properties

1. The union of any collection of open sets is open.

2. The intersection of any finite collection of open sets is open.

3. The intersection of any collection of closed sets is closed.

4. The union of any finite collection of closed sets is closed.

Q:1. The union of any collection of open sets is open.

Proof: Let $\{U_i\}$ be a collection of open sets and let $U = \cup U_i$.

For any $x \in U$, there exists some i such that $x \in U_i$.

Since U_i is open, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq U_i \subseteq U$.

Therefore, U is open.

Q:2. Intersection of any finite collection of open sets is open.

Proof : Let $\{U_1, U_2, ..., U_n\}$ be a finite collection of open sets and let $U = \cap U_i$.

For any $x \in U$, $x \in U_i$ for all i.

Since each U_i is open, there exists $\varepsilon_i > 0$ such that $(x - \varepsilon_i, x + \varepsilon_i) \subseteq U_i$.

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_n\} > 0$.

Then, $(x - \varepsilon, x + \varepsilon) \subseteq U_i$ for all i, so $(x - \varepsilon, x + \varepsilon) \subseteq U$.

Therefore, U is open.

Q:3. The intersection of any collection of closed sets is closed.

Proof : Let $\{F_i\}$ be a collection of closed sets and let $F = \cap F_i$.

Let x be a limit point of F.

Then, x is a limit point of each F_i, since $F \subseteq F_i$ for all i.

Since each F_i is closed, $x \in F_i$ for all i.

Therefore, $x \in F$, so F is closed.

Q:4.The union of any finite collection of closed sets is closed.

Proof : Let $\{F_1, F_2, ..., F_n\}$ be a finite collection of closed sets and let $F = \cup F_i$.

Let x be a limit point of F.

Then, every neighborhood of x contains infinitely many points of F.

Since F is the union of finitely many sets, at least one F_i must contain infinitely many of these points.

Therefore, x is a limit point of F_i, and since F_i is closed, $x \in F_i \subseteq F$.

So, F is closed.

Compactness

Definition: A set S is compact if every open cover of S has a finite sub cover.

1. An open cover of S is a collection of open sets $\{U_i\}$ such that $S \subseteq \cup U_i$.

2. A finite sub cover is a finite subset of $\{U_i\}$ that still covers S.

Examples

1. Closed intervals [a, b] are compact.

2. Finite sets are compact.

Properties

1. Compact sets are closed and bounded.

2. Continuous functions map compact sets to compact sets.

Connectedness

Definition: A set S is connected if it cannot be written as the union of two disjoint non-empty open sets.

1. A set S is disconnected if there exist open sets U and V such that $S \subseteq U \cup V$, $S \cap U \neq \emptyset$, $S \cap V \neq \emptyset$, and $S \cap U \cap V = \emptyset$.

2. Connected sets are "one piece" in some sense.

Examples: 1. Intervals (a, b), [a, b], (a, b], and [a, b) are connected.

2. The set of rational numbers is not connected.

Properties

1. Continuous functions map connected sets to connected sets.

2. Connected sets are useful in understanding the behavior of functions.

Important Theorems

Q:1 Heine-Borel Theorem: A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Proof:(Compactness implies closed and bounded)

1. Let K be a compact subset of $\mathbb{R}$.

2. To show K is bounded: Suppose K is not bounded. Then, consider the open cover $\{(-n, n)\}_{n=1}$^∞. Since K is compact, there exists a finite subcover $\{(-n_1, n_1), ..., (-n_k, n_k)\}$. Let $N = \max\{n_1, ..., n_k\}$. Then, $K \subseteq (-N, N)$, which implies K is bounded.

3. To show K is closed: Let x be a limit point of K. Suppose $x \notin K$. Then, consider the open cover $\{(-\infty, x - 1/n) \cup (x + 1/n, \infty)\}_{n=1}$^∞. Since K is compact, there exists a finite subcover $\{(-\infty, x - 1/N) \cup (x + 1/N, \infty)\}$. This implies $K \cap (x - 1/N, x + 1/N) = \emptyset$, contradicting x being a limit point of K.

(Closed and bounded implies compactness)

1. Let K be a closed and bounded subset of $\mathbb{R}$.

2. Since K is bounded, $K \subseteq [-M, M]$ for some $M > 0$.

3. Suppose $\{U_i\}$ is an open cover of K. If K cannot be covered by finitely many U_i, then at least one of the intervals $[-M, 0]$ or $[0, M]$ cannot be covered by finitely many U_i. Let I_1 be such an interval.

4. Repeat step 3 by dividing I_1 into two equal subintervals. Continue this process to generate a sequence of nested intervals $\{I_n\}$ such that $K \cap I_n$ cannot be covered by finitely many U_i.

5. Since K is closed and bounded, $\cap I_n$ contains a point $x \in K$.

6. Since $\{U_i\}$ covers K, $x \in U_i$ for some i. Since U_i is open, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq U_i$.

7. For sufficiently large n, $I_n \subseteq (x - \varepsilon, x + \varepsilon) \subseteq U_i$, contradicting the assumption that $K \cap I_n$ cannot be covered by finitely many U_i.

The Heine-Borel Theorem provides a characterization of compact sets in $\mathbb{R}$, which is essential in real analysis and topology.

Q:2. Intermediate Value Theorem

Statement : If f is continuous on [a, b] and k is between f(a) and f(b), then there exists $c \in [a, b]$ such that $f(c) = k$.

Proof: Case 1: $f(a) < k < f(b)$

Define a set $S = \{x \in [a, b] : f(x) \leq k\}$. S is non-empty since $a \in S$ (because $f(a) < k$).

S is bounded above by b. By the completeness property of $\mathbb{R}$, S has a least upper bound, say c. We claim $f(c) = k$.

Suppose $f(c) < k$. Then, by continuity of f, there exists $\delta > 0$ such that $f(x) < k$ for all $x \in (c - \delta, c + \delta) \cap [a, b]$. This implies $c + \delta/2 \in S$, contradicting c being an upper bound of S.

Suppose $f(c) > k$. Then, by continuity of f, there exists $\delta > 0$ such that $f(x) > k$ for all $x \in (c - \delta, c + \delta) \cap [a, b]$. This implies $c - \delta/2$ is an upper bound of S, contradicting c being the least upper bound.

Case 2: $f(b) < k < f(a)$

The proof is similar to Case 1, or you can apply Case 1 to the function $-f(x)$.

The Intermediate Value Theorem guarantees the existence of a point c where $f(c) = k$, given the continuity of f on [a, b] and k being between f(a) and f(b). This theorem has numerous applications in analysis and other areas of mathematics.

Question 3 : Prove that a compact subset of ℝ is bounded.

Solution: Let K be a compact subset of ℝ. Suppose K is not bounded. Then, consider the open cover $\{(-n, n)\}_{n=1}^\infty$. Since K is compact, there exists a finite subcover $\{(-n_1, n_1), ..., (-n_k, n_k)\}$. Let $N = \max\{n_1, ..., n_k\}$. Then, $K \subseteq (-N, N)$, which implies K is bounded.

Question 4: Determine whether the set (0, 1) is compact.

Solution : The set (0, 1) is not compact because it is not closed. Consider the sequence $\{1/n\}_{n=1}^\infty \subset (0, 1)$, which converges to $0 \notin (0, 1)$. Alternatively, consider the open cover $\{(1/n, 1)\}_{n=2}^\infty$, which has no finite sub cover.

Question 5: Let $f: [a, b] \to \mathbb{R}$ be continuous. Prove that f attains its maximum and minimum values on [a, b].

Solution: Since f is continuous on [a, b] and [a, b] is compact, f([a, b]) is compact. Therefore, f([a, b]) is closed and bounded. Since f([a, b]) is bounded, it has a least upper bound M and a greatest lower bound m. Since f([a, b]) is closed, $M \in f([a, b])$ and $m \in f([a, b])$. Hence, there exist $x_1, x_2 \in [a, b]$ such that $f(x_1) = M$ and $f(x_2) = m$.

Question 6: Show that the set $[0, 1] \cup [2, 3]$ is compact.

Solution: The set $[0, 1] \cup [2, 3]$ is closed and bounded. By the Heine-Borel Theorem, it is compact.

UNIT-2

Real Sequences

Definition: A real sequence is a function from ℕ (or a subset of ℕ) to ℝ.

A sequence is often denoted as $\{a_n\}$ or $\{a_1, a_2, a_3, ...\}$.

Limit of a Sequence

Definition :A sequence $\{a_n\}$ converges to a limit L if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n \geq N$.

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$.

Examples

1. The sequence $\{1/n\}$ converges to 0 because for any $\varepsilon > 0$, we can choose $N > 1/\varepsilon$.

2. The sequence $\{(-1)^n\}$ does not converge because it oscillates between -1 and 1.

Convergence of a Sequence

Definition: A sequence $\{a_n\}$ is said to converge if there exists a real number L such that $\lim_{n\to\infty} a_n = L$.

Properties

1. Uniqueness of Limit: If a sequence converges, its limit is unique.

2. Boundedness: A convergent sequence is bounded.

3. Algebraic Operations: If $\{a_n\}$ and $\{b_n\}$ converge to A and B, respectively, then $\{a_n + b_n\}$ converges to A + B, $\{a_n b_n\}$ converges to AB, and $\{a_n/b_n\}$ converges to A/B (if $B \neq 0$).

Examples of Convergent Sequences

1. $\{1/n\}$ converges to 0.

2. $\{(n+1)/n\}$ converges to 1.

3. $\{2^n/n!\}$ converges to 0.

Examples of Divergent Sequences

1. $\{(-1)^n\}$ diverges because it oscillates.

2. $\{n\}$ diverges to ∞.

3. $\{2^n\}$ diverges to ∞.

Understanding the limit and convergence of sequences is fundamental in real analysis, as it lays the groundwork for more advanced topics like series, continuity, and calculus.

Monotonic Sequences

Definition: A sequence $\{a_n\}$ is said to be monotonic if it is either monotonically increasing or monotonically decreasing.

Types of Monotonic Sequences

1. Monotonically Increasing: A sequence $\{a_n\}$ is monotonically increasing if $a_n \leq a_{n+1}$ for all n.

2. Monotonically Decreasing: A sequence $\{a_n\}$ is monotonically decreasing if $a_n \geq a_{n+1}$ for all n.

3. Strictly Increasing: A sequence $\{a_n\}$ is strictly increasing if $a_n < a_{n+1}$ for all n.

4. Strictly Decreasing: A sequence $\{a_n\}$ is strictly decreasing if $a_n > a_{n+1}$ for all n.

Examples

Monotonically Increasing

1. {1, 2, 3, 4, ...} (strictly increasing)

2. {1, 1, 2, 2, 3, 3, ...} (monotonically increasing)

Monotonically Decreasing

1. {4, 3, 2, 1, ...} (strictly decreasing)

2. {4, 4, 3, 3, 2, 2, ...} (monotonically decreasing)

Properties

1. A monotonic sequence is convergent if and only if it is bounded.

2. A bounded monotonic sequence converges to its supremum (if increasing) or infimum (if decreasing).

Important Result

Monotone Convergence Theorem

If $\{a_n\}$ is a monotonic sequence that is bounded, then $\{a_n\}$ converges.

Proof: Case: 1: $\{a_n\}$ is monotonically increasing. Since $\{a_n\}$ is bounded, there exists $M \in \mathbb{R}$ such that $a_n \leq M$ for all n. Let $S = \{a_n : n \in \mathbb{N}\}$. Then, S is a non-empty subset of $\mathbb{R}$ that is bounded above.By the completeness property of $\mathbb{R}$, S has a least upper bound, say L.We claim that $\lim_{n \to \infty} a_n = L$.

Proof of $\lim_{n \to \infty} a_n = L$

For any $\varepsilon > 0$, $L - \varepsilon$ is not an upper bound of S.Therefore, there exists $N \in \mathbb{N}$ such that $a_n > L - \varepsilon$ for some N. Since $\{a_n\}$ is monotonically increasing, $a_n \geq a_n > L - \varepsilon$ for all $n \geq N$. Also, $a_n \leq L < L + \varepsilon$ for all n. Hence, $|a_n - L| < \varepsilon$ for all $n \geq N$.

Case 2: $\{a_n\}$ is monotonically decreasing.The proof is similar to Case 1, or you can apply Case 1 to the sequence $\{-a_n\}$.

Example Consider the sequence $\{a_n\}$ defined by $a_n = 1 - 1/n$. This sequence is monotonically increasing and bounded above by 1. By the Monotone Convergence Theorem, $\{a_n\}$ converges to its least upper bound, which is 1.

Question 1

Prove that the sequence $\{a_n\}$ defined by $a_n = 1 - 1/n$ converges.

Solution: The sequence $\{a_n\}$ is monotonically increasing because $a_{n+1} - a_n = 1/(n+1)n > 0$.The sequence $\{a_n\}$ is bounded above by 1 because $a_n = 1 - 1/n < 1$ for all n.By the Monotone Convergence Theorem, $\{a_n\}$ converges.

Question 2

Show that the sequence $\{a_n\}$ defined by $a_n = 1 + 1/2 + ... + 1/n$ does not converge.

Solution: The sequence $\{a_n\}$ is monotonically increasing because $a_{n+1} - a_n = 1/(n+1) > 0$.However, $\{a_n\}$ is not bounded above (it is the harmonic series, which diverges). Therefore, $\{a_n\}$ does not converge.

Question 3

Let $\{a_n\}$ be a sequence defined by $a_1 = 1$ and $a_{n+1} = \sqrt{(2 + a_n)}$. Show that $\{a_n\}$ converges.

Solution: We can show that $\{a_n\}$ is monotonically increasing and bounded above by 2. By the Monotone Convergence Theorem, $\{a_n\}$ converges.

Question 4

Find the limit of the sequence $\{a_n\}$ defined by $a_n = (1 + 1/n)^n$.

Solution: The sequence $\{a_n\}$ is monotonically increasing and bounded above by e.

By the Monotone Convergence Theorem, $\{a_n\}$ converges to e.

Cauchy's General Principle of Convergence

Statement:A sequence $\{a_n\}$ converges if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon$ for all $n, m \geq N$.

Proof: (Convergence implies Cauchy condition): Suppose $\{a_n\}$ converges to L. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon/2$ for all $n \geq N$. Then, for all $n, m \geq N$, $|a_n - a_m| \leq |a_n - L| + |L - a_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

(Cauchy condition implies convergence)

Suppose $\{a_n\}$ satisfies the Cauchy condition. Then, $\{a_n\}$ is bounded (take $\varepsilon = 1$, then $|a_n - a_m| < 1$ for all $n, m \geq N$). By the Bolzano-Weierstrass theorem, $\{a_n\}$ has a convergent subsequence $\{a_{nk}\}$ converging to L. We claim that $\{a_n\}$ converges to L.

Proof of $\lim_{n \to \infty} a_n = L$

For any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon/2$ for all $n, m \geq N_1$.There exists $N_2 \in \mathbb{N}$ such that $|a_{nk} - L| < \varepsilon/2$ for all $k \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$, $|a_n - L| \leq |a_n - a_{nk}| + |a_{nk} - L| < \varepsilon$.

Example

Consider the sequence $\{a_n\}$ defined by $a_n = 1 + 1/2! + ... + 1/n!$. This sequence satisfies the Cauchy condition and converges.

Questions with Answers

Q1: Prove that the sequence $\{a_n\}$ defined by $a_n = 1/n$ satisfies the Cauchy condition.

Solution : For any $\varepsilon > 0$, choose $N > 2/\varepsilon$. Then, for all $n, m \geq N$, $|a_n - a_m| \leq |1/n| + |1/m| < \varepsilon$.

Q2: Show that the sequence $\{a_n\}$ defined by $a_n = (-1)^n$ does not satisfy the Cauchy condition.

Solution: For any N, $|a_n - a_m| = 2$ for $n = N$ and $m = N + 1$ if N is even.

Cauchy's General Principle of Convergence provides a necessary and sufficient condition for convergence, which is useful in determining the convergence of sequences.

Continuous Functions on Closed Intervals

Properties

Q: Boundedness: If f is continuous on [a, b], then f is bounded on [a, b].

Proof: Assume f is not bounded.Suppose f is continuous on [a, b] but not bounded.

Then, for every $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $|f(x_n)| > n$.

Constructing a contradiction

1. Since $\{x_n\}$ is a sequence in [a, b], which is closed and bounded, $\{x_n\}$ has a convergent subsequence $\{x_{nk}\}$ converging to some $x \in [a, b]$.
2. Since f is continuous at x, $f(x_{nk}) \to f(x)$ as $k \to \infty$.
3. However, $|f(x_{nk})| > n_k$ for all k, which implies $f(x_{nk})$ does not converge.

Conclusion

1. This contradicts the fact that $f(x_{nk}) \to f(x)$.

2. Therefore, our assumption that f is not bounded must be false.

Final result

f is bounded on [a, b].

The Boundedness Theorem guarantees that a continuous function on a closed interval is bounded, which is a fundamental property in real analysis.

Q: Extreme Value Theorem: If f is continuous on [a, b], then f attains its maximum and minimum values on [a, b].

Proof

Step 1: Boundedness: Since f is continuous on [a, b], f is bounded on [a, b] (by the Boundedness Theorem). Let $M = \sup\{f(x) : x \in [a, b]\}$ and $m = \inf\{f(x) : x \in [a, b]\}$. We claim that there exists $x_1 \in [a, b]$ such that $f(x_1) = M$. Suppose not. Then, $f(x) < M$ for all $x \in [a, b]$. Consider the function $g(x) = 1/(M - f(x))$, which is continuous on [a, b]. Since g(x) is continuous on [a, b], g(x) is bounded on [a, b]. However, since f(x) can be arbitrarily close to M, g(x) can be arbitrarily large, contradicting the bounded ness of g(x).Attaining the minimum value

The proof is similar to Step 2, or you can apply Step 2 to the function -f(x).

Conclusion: f attains its maximum and minimum values on [a, b].The Extreme Value Theorem guarantees that a continuous function on a closed interval attains its maximum and inimum values, which is a fundamental property in real analysis and optimization.

Q: **Intermediate Value Theorem: If f is continuous on [a, b] and k is between f(a) and f(b), then there exists $c \in [a, b]$ such that $f(c) = k$.**

Proof:

Without loss of generality, assume $f(a) < k < f(b)$. Let $S = \{x \in [a, b] : f(x) < k\}$.

S is non-empty since $a \in S$ (because $f(a) < k$). S is bounded above by b.

By the completeness property of $\mathbb{R}$, S has a least upper bound, say c.

Claim: $f(c) = k$

Suppose $f(c) < k$. Then, there exists $\delta > 0$ such that $f(x) < k$ for all $x \in (c - \delta, c + \delta) \cap [a, b]$. This implies $c + \delta/2 \in S$, contradicting the fact that c is an upper bound of S.

Suppose $f(c) > k$, Then, there exists $\delta > 0$ such that $f(x) > k$ for all $x \in (c - \delta, c + \delta) \cap [a, b]$. This implies $c - \delta$ is an upper bound of S, contradicting the fact that c is the least upper bound of S.

Conclusion $f(c) = k$.

The Intermediate Value Theorem guarantees that a continuous function on a closed interval takes on all values between its endpoint values, which has numerous applications in mathematics and science.

Q1: Prove that the function $f(x) = x^2$ is bounded on [0, 1].

A1: Since f is continuous on [0, 1] and [0, 1] is closed, f is bounded on [0, 1]. Specifically, $0 \leq f(x) \leq 1$ for all $x \in [0, 1]$.

Q2: Show that the function $f(x) = x^3$ attains its maximum and minimum values on [-1, 1].

A2: Since f is continuous on [-1, 1] and [-1, 1] is closed, f attains its maximum and minimum values on [-1, 1]. The minimum value is $f(-1) = -1$, and the maximum value is $f(1) = 1$.

Q3: Prove that the equation $x^3 + x - 1 = 0$ has a root in [0, 1].

A3: Let $f(x) = x^3 + x - 1$. Then, $f(0) = -1$ and $f(1) = 1$. By the Intermediate Value Theorem, there exists $c \in [0, 1]$ such that $f(c) = 0$.

Q4: Show that the function $f(x) = 1/x$ is not bounded on (0, 1).

A4: Since f is not defined at $x = 0$, and (0, 1) is not closed, f is not bounded on (0, 1). Specifically, $f(x) \to \infty$ as $x \to 0^+$.

These questions demonstrate the application of properties of continuous functions on closed intervals, including boundedness, extreme values, and intermediate values.

Q:5What does the Intermediate Value Theorem state?

A5: The Intermediate Value Theorem states that if f is continuous on [a, b] and k is between f(a) and f(b), then there exists $c \in [a, b]$ such that f(c) = k.

Q6: Is the function f(x) = 1/x continuous on [0, 1]?

A6: No, the function f(x) = 1/x is not continuous on [0, 1] because it is not defined at x = 0.

Q7: Does the function $f(x) = x^2$ attain its maximum and minimum values on [-1, 1]?

A7: Yes, the function $f(x) = x^2$ attains its maximum value 1 at $x = \pm 1$ and its minimum value 0 at x = 0.

Q8: Prove that the equation $x^3 + x - 1 = 0$ has a root in [0, 1].

A8: Let $f(x) = x^3 + x - 1$. Then, f(0) = -1 and f(1) = 1. By the Intermediate Value Theorem, there exists $c \in [0, 1]$ such that f(c) = 0.

UNITL-III

Exact Differential Equations

A differential equation of the form M(x, y)dx + N(x, y)dy = 0 is said to be exact if there exists a function f(x, y) such that $\partial f/\partial x = M$ and $\partial f/\partial y = N$.

Condition for Exactness

M(x, y)dx + N(x, y)dy = 0 is exact if and only if $\partial M/\partial y = \partial N/\partial x$.

Solution of Exact Differential Equations

If M(x, y)dx + N(x, y)dy = 0 is exact, then the solution is given by $f(x, y) = \int M(x, y)dx + \int N(x, y)dy$ (terms not containing x in N are considered).

Q:1 $(2xy + y^2)dx + (x^2 + 2xy)dy = 0$

Solution: Step 1: Check for exactness

$\partial M/\partial y = \partial(2xy + y^2)/\partial y = 2x + 2y$

$\partial N/\partial x = \partial(x^2 + 2xy)/\partial x = 2x + 2y$

Since $\partial M/\partial y = \partial N/\partial x$, the equation is exact.

Step 2: Find the function f(x, y)

$\partial f/\partial x = M = 2xy + y^2$

$\partial f/\partial y = N = x^2 + 2xy$

Step 3: Integrate M with respect to x

$f(x, y) = \int(2xy + y^2)dx = x^2y + xy^2 + g(y)$

Step 4: Differentiate f(x, y) with respect to y and equate to N

$\partial f/\partial y = x^2 + 2xy + g'(y) = x^2 + 2xy$

$g'(y) = 0$

Step 5: Solve for g(y)

g(y) = C (constant)

Step 6: Write the solution

$x^2y + xy^2 = C$

The solution to the differential equation $(2xy + y^2)dx + (x^2 + 2xy)dy = 0$ is $x^2y + xy^2 = C$.

Example 5 : Solve $p = \tan\left(x - \frac{p}{1+p^2}\right)$.

Solution : The given equation can be written as

$$x = \tan^{-1}p + \frac{p}{1+p^2} \qquad \text{..... (23)}$$

Differentiating Eqn. (23) w.r. to y, we get

$$\frac{1}{p} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{(1+p^2)-p(2p)}{(1+p^2)^2}\frac{dp}{dy}$$

$$= \frac{1+p^2+1+p^2-2p^2}{(1+p^2)^2}\frac{dp}{dy}$$

$$= \frac{2}{(1+p^2)^2}\frac{dp}{dy}$$

$$\Rightarrow dy = \frac{2p}{(1+p^2)^2}dp \qquad \text{..... (24)}$$

Note that Eqn. (24) is in variable separable form.

Integrating Eqn. (24), we get

$$y = c - \frac{1}{1+p^2}, \qquad \text{..... (25)}$$

c being an arbitrary constant.

It is not possible to eliminate p between Eqns. (23) and (25). Thus, Eqns. (23) and (25) together constitute the solution of the given equation in terms of parameter p.

Example 2 : Solve $p^3(x+2y) + 3p^2(x+y) + (y+2x)p = 0$

Solution : The given equation is equivalent to

$p[p^2(x+2y) + 3p(x+y) + (y+2x)] = 0$

$\Rightarrow p[p^2(x+2y) + p\{(y+2x)+(x+2y)\} + (y+2x)] = 0$

$\Rightarrow p(p+1)[(x+2y)p + (y+2x)] = 0$

Its component equations are

$p = 0,\ p+1 = 0,\ (x+2y)p + (y+2x) = 0$

Now $p = 0 \Rightarrow \dfrac{dy}{dx} = 0$, which has the solution

$y = c$ ..... (5)

Now $p+1 = 0 \Rightarrow \dfrac{dy}{dx} + 1 = 0$,

i.e., $dy + dx = 0$,

which has the solution

$y+x = c$ ..... (6)

Further, $(x+2y)p + (y+2x) = 0$

$\Rightarrow (x+2y)dy + (y+2x)dx = 0$,

$\Rightarrow d(xy+x^2+y^2) = 0$,

which has the solution

$xy + x^2 + y^2 = c$ ..... (7)

Therefore, the general solution of the given equation, from Eqns. (5), (6) and (7), is

$(y-c).(y+x-c).(xy+x^2+y^2-c) = 0$.

Example 7 : Solve $y = 2p + 3p^2$

Solution : We have

$y = 2p + 3p^2$ ..... (32)

which is already in the form $y = F(p)$. Following the method discussed in Sec. 4.3.1, we differentiate it w.r.t x, so that

$$p = 2\frac{dp}{dx} + 6p\frac{dp}{dx}$$

or $\dfrac{p}{2+6p} = \dfrac{dp}{dx}$

Here variable are separable and we have

$$dx = \left(\frac{2}{p} + 6\right)dp$$

Integrating, we get

$x = 6p + 2\ln|p| + c$. ..... (33)

c being an arbitrary constant.

Since it is not possible to eliminate p from Eqns. (32) and (33), these equations together yield the required solution in terms of the parameter p.

Example 10 : Solve $p^2 - 2xp + 1 = 0$

Solution : The given equation is

$p^2 - 2xp + 1 = 0$

Solving for p, we get

$$p = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$$

$\therefore$ Either $p = x + \sqrt{x^2-1}$ or $p = x - \sqrt{x^2-1}$

Now $p = x + \sqrt{x^2-1}$, on integration yields

$$y = \frac{x^2}{2} + \frac{x\sqrt{x^2-1}}{2} - \frac{1}{2}\ln|x + \sqrt{x^2-1}| + c,$$

c being an arbitrary constant.

Similarly, $p = x - \sqrt{x^2-1}$ yields

$$y = \frac{x^2}{2} - \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\ln|x + \sqrt{x^2-1}| + c,$$

Hence, the general solution of the given equation is

$$[x^2 + x\sqrt{x^2-1} - \ln|x + \sqrt{x^2-1}| - 2y + c_1][x^2 - x\sqrt{x^2-1} + \ln|x + \sqrt{x^2-1}| - 2y + c_1] = 0.$$

where $c_1 = 2c$ is an arbitrary constant.

Example 12 : Solve $(y')^2 + 4xy' - 4y = 0$ ..... (49)

Solution : With $p = y'$, Eqn. (49) can be rewritten as

$$y = px + \frac{1}{4}p^2, \qquad \text{.... (50)}$$

which is in the Clairaut's form. Differentiating Eqn. (50) w.r. to x, we get

$$p = p + p'x + \frac{p}{2}p'$$

$$\Rightarrow p'(x + \frac{p}{2}) = 0$$

Then either $p' = 0$ which gives $p = c$ (a constant) ..... (51)

or $x + \frac{p}{2} = 0$ (52)

From Eqns (50) and (51), we obtain

$$y = cx + \frac{c^2}{4}$$

as the solution of Eqn. (50). Eliminating p from Eqns.(50) and (52), we get

$$y = x(-2x) + \frac{1}{4}(-2x)^2,$$

i.e., $y(x) = -x^2$,

which contains no arbitrary constant. Since this value of y satisfies Eqn. (50), it is the singular solution of Eqn. (50).

Ex 3. *Solve* : $\frac{d^3y}{dx^3}-8y=0.$

The auxiliary equation is $m^3-8=0$, the roots of which are $m=2$ and $m=-1\pm i\sqrt{3}$.

Hence the solution is

$$y=e^{-x}\,(c_1 \cos \sqrt{3}\,x+c_2 \sin \sqrt{3}\,x)+c_3\, e^{2x}.$$

Ex. 4. *Solve* : $(D^3-2D^2+D)\,y=e^{-x}$.

Here the auxiliary equation is

$$m^3-2m^2+m=0$$

$\therefore$ $m=0, m=1$ repeated twice.

$\therefore$ The complementary function is

$$c_1+(c_2+c_3\,x)\;e^{x}.$$

The particular integral

$$=\frac{1}{D^3-2D^2+D}\,e^{-x}$$

$$=\left[\frac{1}{D}-\frac{1}{D-1}+\frac{1}{(D-1)^2}\right]e^{-x}$$

$$=\frac{1}{D}\,e^{-x}-\frac{1}{D-1}\,e^{-x}+\frac{1}{(D-1)(D-1)}\,e^{-x}$$

$$=\int e^{-x}\,dx-e^{x}\int e^{-x}\,e^{-x}\,dx+\frac{1}{D-1}\left[e^{x}\int e^{-x}e^{-x}\,dx\right]$$

$$=e^{-x}+\tfrac{1}{2}\,e^{-x}-\tfrac{1}{2}\frac{1}{D-1}\;e^{-x}$$

$$=-e^{-x}+\tfrac{1}{2}\,e^{-x}+\tfrac{1}{4}\,e^{-x},\text{ since }\frac{1}{D-1}\,e^{-x}=-\tfrac{1}{2}\,e^{-x}.$$

$$=-\tfrac{1}{4}\,e^{-x}$$

Ex. 5. *Solve* : $\frac{d^2y}{dx^2}+n^2y=\sec nx.$ (*Nagpur* 195

The auxiliary equation is

$$m^2+n^2=0, \text{ or } m=\pm i\,n,$$

$\therefore$ The complementary function is

$$c_1 \cos nx+c_2 \sin nx.$$

The particular integral

$$=\frac{1}{D^2+n^2} \sec nx=\frac{1}{(D+in)(D-in)} \sec nx$$

$$=\frac{1}{2in}\left\{\frac{1}{D-in}-\frac{1}{D+in}\right\} \sec nx$$

Now $\frac{1}{D-in} \sec nx=e^{inx} \int \frac{e^{-inx}}{\cos nx}\,dx$

$$=e^{inx} \int \frac{\cos nx-i \sin nx}{\cos nx}\,dx$$

$$=e^{inx}\{ x+i\,(1/n) \log \cos nx \}$$

Similarly $\frac{1}{D+in} \sec nx=e^{-inx}\{ x-i\,(1/n) \log \cos nx \}$

$\therefore$ $P.\,I.=(1/n)\{ x \sin nx+(1/n)(\log \cos nx) \cos nx \}$

$\therefore$ The complete solution is

$$y=c_1 \cos nx+c_2 \sin nx+\frac{x \sin nx}{n}+\frac{\cos nx \log \cos nx}{n^2}.$$

Ex. 6. *Solve* : $\frac{d^3y}{dx^3}+y=(e^x+1)^2.$

Written in symbolic form, this equation becomes

$$(D^3+1)\,y=e^{2x}+2e^x+1,$$

or $(D+1)(D^2-D+1)\,y=e^{2x}+2e^x+1.$

Here the roots of $f(m)=0$ are -1, $\frac{1}{2}(1 \pm i\sqrt{3})$ ence the complementary function is

$$c_1 e^{-x}+e^{x/2}(c_2 \cos \tfrac{1}{2}\sqrt{3}\,x+c_3 \sin \tfrac{1}{2}\sqrt{3}\,x)$$

The particular integral

$$=\frac{1}{D^3+1}(e^{2x}+2e^x+e^{0.x})$$

$$=\frac{1}{2^3+1}e^{2x}+2\cdot\frac{1}{1^3+1}e^x+\frac{1}{0+1}e^{0.x}$$

$$=\tfrac{1}{9}e^{2x}+e^x+1$$

Hence the complete solution is

$$y=c_1 e^{-x}+e^{x/2}(c_2 \cos \tfrac{1}{2}\sqrt{3}\,x+c_3 \sin \tfrac{1}{2}\sqrt{3}\,x)+\tfrac{1}{9}e^{2x}+e^x+1.$$

Ex. 7. *Solve :* $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+y=a\cos 2x.$

Here $f(D)=D^2-4D+1$

$\therefore$ The roots of $f(m)=0$ are $2\pm\sqrt{3}$.

Hence the complementary function is

$$e^{2x}\left(c_1 e^{\sqrt{3}x}+c_2 e^{-\sqrt{3}x}\right)$$

The particular integral

$$=\frac{1}{D^2-4D+1}a\cos 2x=a\cdot\frac{1}{(-2^2)-4D+1}\cos 2x$$

$$=-a\cdot\frac{1}{4D+3}\cos 2x=-a\cdot\frac{(4D-3)}{(4D+3)(4D-3)}\cos 2x,$$

since $(4D-3)$ and $\frac{1}{4D-3}$ are inverse operators ;

$$=-a\cdot\frac{(4D-3)}{16D^2-9}\cos 2x=-a\cdot\frac{4D-3}{16(-2^2)-9}\cos 2x$$

$$=(a/73)(4D-3)\cos 2x=(a/73)\cdot(-4\cdot 2\cdot\sin 2x-3\cos 2x)$$

$$=-(a/73)(8\sin 2x+3\cos 2x).$$

$\therefore$ The complete solution is

$$e^{2x}\left(c_1 e^{\sqrt{3}x}+c_2 e^{-\sqrt{3}x}\right)-(a/73)(8\sin 2x+3\cos 2x).$$

Ex. 9. *Solve :* $\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+4\frac{dy}{dx}-2y=e^x+\cos x$

(*Agra* 1960)

The auxiliary equation is

$$m^3-3m^2+4m-2=0 \quad\text{or}\quad m=1,\ 1\pm i$$

Hence the *C. F.* is $c_1 e^x+c_2 e^x\cos(x+\alpha)$.

$$\text{The } P.I.=\frac{1}{(D-1)(D^2-2D+2)}(e^x+\cos x)$$

$$=\frac{1}{(D-1)(1-2+2)}e^x+\frac{1}{(-1-2D+2)(D-1)}\cos x$$

$$=e^x.\ \frac{1}{(D+1-1)}\cdot 1+\frac{1}{-2D^2+3D-1}\cos x$$

$$=x\ e^x+\frac{1}{3D+1}\cos x=x\ e^x+\frac{3D-1}{9D^2-1}\cos x$$

$$=x\ e^x+\frac{3D-1}{-9-1}\cos x=x\ e^x-\frac{1}{10}(3D-1)\cos x$$

$$=x\ e^x+\frac{1}{10}(3\sin x+\cos x).$$

Hence the complete solution is

$$y=c_1 e^x+c_2 e^x\cos(x+\alpha)+x\ e^x+\frac{1}{10}(3\sin x+\cos x).$$

Ex. 10. *Solve* : $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=x\sin x.$

The complementary function is $(c_1+c_2x)e^x$.

The particular integral

$$=\frac{1}{D^2-2D+1}\,x\sin x$$

$$=x\cdot\frac{1}{D^2-2D+1}\sin x-2(D-1)\cdot\frac{1}{(D^2-2D+1)^2}\sin x$$

$$=x\cdot\left(-\frac{1}{2D}\sin x\right)-2(D-1)\frac{1}{4D^2}\sin x.$$

$$=\tfrac{1}{2}x\cos x+\tfrac{1}{2}(D-1)\sin x=\tfrac{1}{2}(x\cos x+\cos x-\sin x).$$

$\therefore$ The complete solution is

$$y=(c_1+c_2x)e^x+\tfrac{1}{2}(x\cos x+\cos x-\sin x).$$

Ex. 11. *Solve* : $\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}+2\frac{dy}{dx}=x^2.$

The auxiliary equation is $m^3+3m^2+2m=0$; its roots are $m=0,-1,-2.$

Hence the complementary function is $c_1+c_3e^{-x}+c_3e^{-2x}$

The particular integral

$$=\frac{1}{D^3+3D^2+2D}x^2=\frac{1}{2D}\cdot\frac{1}{\left(1+\frac{D^2+3D}{2}\right)}x^2$$

$$=\frac{1}{2D}\left(1+\frac{D^2+3D}{2}\right)^{-1}x^2$$

$$=\frac{1}{2D}\left[1-\frac{D^2+3D}{2}+\left(\frac{D^2+3D}{2}\right)^2-\dots\dots\right]x^2$$

$$=\frac{1}{2D}\left[1-\frac{D^2}{2}-\frac{3D}{2}+\tfrac{1}{4}(D^4+6D^3+9D^2)+\dots\dots\right]x^2$$

$$=\frac{1}{2D}\left(1-\frac{3D}{2}+\frac{7}{4}D^2+\dots\dots\right)x^2$$

$$=\frac{1}{2}\left(\frac{1}{D}-\frac{3}{2}+\frac{7}{4}D\right)x^2$$

$$=\frac{1}{2}\left[\frac{1}{3}x^3-\frac{3}{2}x^2+\frac{7}{4}\cdot 2x\right]$$

$$=\frac{1}{12}(2x^3-9x^2+21x).$$

$\therefore$ The complete solution is

$$y=c_1+c_2e^{-x}+c_3e^{-2x}+\frac{x}{12}(2x^2-9x+21)$$

Ex. 12. *Solve :* $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=x^2 e^{3x}$.

The auxiliary equation is $m^2-2m+1=0$. Its roots are $m=1$, repeated twice.

$\therefore$ The complementary function is $(c_1+c_2 x)\, e^x$

The particular integral

$$=\frac{1}{D^2-2D+1}\, x^2 e^{3x}$$

$$=e^{3x}\cdot\frac{1}{(D+3)^2-2\,(D+3)+1}\, x^2, \text{ by the formula of this article ;}$$

$$=e^{3x}\cdot\frac{1}{D^2+4D+4}\, x^2=\frac{e^{3x}}{4}\cdot\left(1+\frac{D}{2}\right)^{-2} x^2$$

$$=\frac{e^{3x}}{4}\left(1-D+3.\ \frac{D^2}{4}-\ldots\ldots\right)x^2$$

$$=\frac{e^{3x}}{4}\left(x^2-2x+\frac{3}{4}\,.\,2\right)=\frac{e^{3x}}{8}\,(2x^2-4x+3)$$

$\therefore$ The complete solution is

$$y=(c_1+c_2\,x)\,e^x+\tfrac{1}{8}e^{3x}\,(2x^2-4x+3).$$

Ex. 13. *Solve :* $\frac{d^2y}{dx^2}+4\,\frac{dy}{dx}+4y=2\ \sinh 2x.$

The equation, written in the symbolic form, is

$$(D^2+4D+4)\ y=e^{2x}-e^{-2x}.$$

$\therefore$ The complementary function is $(c_1+c_2\,x)\ e^{-2x}$.

$$\text{The } P.I.=\frac{1}{D^2+4D+4}\ (e^{2x}-e^{-2x})$$

$$=\frac{1}{D^2+4D+4}\ e^{2x}-\frac{1}{D^2+4D+4}\ e^{-2x}.\ 1$$

$$=\frac{1}{2^2+4\cdot2+4}\ e^{2x}-e^{-2x}\frac{1}{[(D-2)^2+4(D-2)+4\ ]}\cdot 1$$

$$=\frac{1}{16}\ e^{2x}-e^{2x}\left(\frac{1}{D^2}\right)1=\frac{e^{2x}}{16}-e^{-2x}\ \frac{1}{D}\int 1\ dx$$

$$=\tfrac{1}{16}\ e^{2x}-e^{-2x}\int x\ dx=\tfrac{1}{16}\ e^{2x}-\tfrac{1}{2}\ x^2\ e^{-2x}$$

$\therefore$ The complete solution is

$$y=(c_1+c_2\ x)\ e^{-2x}+\tfrac{1}{16}\ e^{2x}-\tfrac{1}{2}\ x^2\ e^{-2x}$$

EXERCISE III (II)

Ex. 14. *Solve* : $\frac{d^2y}{dx^2}+a^2y=\cos ax$.

The complementary function is $c_1 \sin ax+c_2 \cos ax$

The $P.I.=\frac{1}{D^2+a^2} \cos ax$

$$=\text{Real part of } \frac{1}{D^2+a^2} (\cos ax+i \sin ax)$$

$$=\text{Real part of } \frac{1}{D^2+a^2} e^{iax}$$

But $\frac{1}{D^2+a^2} e^{iax}=\frac{1}{D-ia}\left(\frac{1}{D+ia} e^{iax}\right)$

$$=\frac{1}{D-ia}\left(\frac{e^{iax}}{2ia}\right), \qquad \text{(by Art. 3·7)}$$

$$=\frac{1}{2ia} \cdot e^{iax}, \frac{1}{(D+ia)-ia} \cdot 1$$

$$=\frac{1}{2ia} e^{iax}. x=-\frac{ix}{2a} (\cos ax+i \sin ax)$$

$\therefore$ The real part$=\frac{x}{2a} \sin ax$.

$\therefore$ The complete solution is

$$y=c_1 \sin ax+c_2 \cos ax+\frac{x}{2a} \sin ax$$

[On equating the imaginary part we could get

$$\frac{1}{D^2+a^2} \sin ax=-\frac{x}{2a} \cos ax \;]$$

Ex. 14. *Solve* : $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=x \sin x$.

The complementary function is easily found out as $(c_1+c_2 x) e^x$.

The particular integral

$$=\frac{1}{D^2-2D+1} x \sin x$$

$$=\text{The imaginary part of } \frac{1}{D^2-2D+1} x (\cos x+i \sin x)$$

$$=\text{The imaginary part of } \frac{1}{(D-1)^2} x e^{ix}$$

$$= e^{ix} \frac{1}{(D+i-1)^2} x$$

$$= \frac{e^{ix}}{k^2} \left\{ \left(1+\frac{D}{x} \right)^{-2} x \right\}, \text{ on putting } i-1=k$$

$$= \frac{e^{ix}}{k^2} \left\{ \left(1-\frac{2D}{k}+\cdots\cdots \right) x \right\}$$

$$= \frac{e^{ix}}{k^2} \left(x-\frac{2}{k} \right).$$

$$= \left(\frac{\cos x+i \sin x}{-2i} \right) \left[x+\frac{2}{2i}(i-1) \right], \therefore k^2=-2i$$

$$= \left(\frac{i \cos x}{2}-\frac{\sin x}{2} \right) (x+1+i)$$

$\therefore$ The imaginary part

$= \frac{1}{2} (x \cos x+\cos x-\sin x)$.

$\therefore$ The complete solution is

$y=(c_1+c_2 x) e^x+\frac{1}{2} (x \cos x+\cos x-\sin x)$.

UNIT_4

Ex. 1. *Solve* :

$$x^4 \frac{d^3y}{dx^3}+2x^3 \frac{d^2y}{dx^2}-x^2\frac{dy}{dx}+xy=1.$$

(*Bombay* 1961, *Agra* 1955, *Mysore* 1948)

Dividing both sides of the equation by x

$$x^3 \frac{d^3y}{dx^3}+2x^2 \frac{d^2y}{dx^2}-x \frac{dy}{dx}+y=\frac{1}{x}.$$

On changing the independent variable by putting $x=e^z$, this ation is reduced to

$$[D(D-1)(D-2)+2D(D-1)-D+1] y=e^{-z},$$

$$(D-1)^2 (D+1) y=e^{-z}.$$

The $C.F.=(c_1+c_2 z) e^z+c_3 e^{-z}$

$=(c_1+c_2 \log x) x+c_3 x^{-1}$

$$P.I.=\frac{1}{(D-1)^2 (D+1)} \cdot e^{-z}=\frac{1}{D+1} \cdot \frac{1}{(D-1)^2} e^{-z}$$

$$=\frac{1}{D+1} \frac{e^{-z}}{4}=\frac{e^{-z}}{4} \frac{1}{D} \cdot 1=\frac{e^{-z} \cdot z}{4}=\frac{x^{-1} \log x}{4}$$

Hence the complete integral is

$y=(c_1+c_2 \log x) x+c_3 x^{-1}+\frac{1}{4} x^{-1} \log x.$

Ex. 2. *Solve :* $x^2 \dfrac{d^2y}{dx^2} - 2y = x^2 + \dfrac{1}{x}$. (*Jaipur* **1951**)

On changing the independent variable by putting $x = e^z$, the equation is reduced to $D(D-1)\, y - 2y = e^{2z} + e^{-z}$

i.e. $(D^2 - D - 2)\, y = e^{2z} + e^{-z}$

The auxiliary equation is : $m^2 - m - 2 = 0$ giving $m = 2$ or -1

$\therefore$ *C. F.* is $c_1 e^{2z} + c_2 e^{-z} = c_1 x^2 + c_2 x^{-1}$

$$P.\ I. = \frac{1}{D^2 - D - 2}\, (e^{2z} + e^{-z})$$

$$= \frac{1}{(D-2)(D+1)}\, e^{2z} + \frac{1}{(D-2)(D+1)}\, e^{-z}$$

$$= \frac{1}{3(D-2)}\, e^{2z} - \frac{1}{3\,(D+1)}\, e^{-z}$$

$$= \frac{1}{3}\, e^{2z} \cdot \frac{1}{D} \cdot 1 - \frac{1}{3}\, e^{-z} \cdot \frac{1}{D} \cdot 1$$

$$= \tfrac{1}{3}\, z\, (e^{2z} - e^{-z}) = \tfrac{1}{3} \log x\, (x^2 - 1/x)$$

Hence the complete integral is

$$y = c_1 x^2 + c_2 x^{-1} + \tfrac{1}{3} \log x . (x^2 - 1/x)$$

Ex. 3. *Solve :* $x^3 \frac{d^3y}{dx^3} - x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = x^2 + 3x.$

(*Raj.* 1955)

On changing the independent variable by putting $x=e^z$, the equation reduces to

$$\{ D(D-1)(D-2) - D(D-1) + 2D - 2 \} y = e^{2z} + 3e^z$$

or $\quad (D^3 - 4D^2 + 5D - 2)\, y = e^{2z} + 3e^z$

or $\quad (D-1)^2 (D-2)\, y = e^{2z} + 3e^z$

The $C.F. = (c_1 + c_2 z)\, e^z + c_3\, e^{2z} = (c_1 + c_2 \log x)\, x + c_3 x^2.$

Ex. 7. *Solve :*

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x} \right).$$

(*Delhi* 1961, *Agra* 1952)

Here $f(\theta)$ is $\theta(\theta-1)(\theta-2) + 2\theta(\theta-1) + 2$, which reduces to

$$(\theta+1)(\theta^2 - 2\theta + 2),$$

the roots of which are $-1,\ 1 \pm i$.

Hence $\quad C.F. = c_1 x^{-1} + x\,(c_2 \cos \log x + c_3 \sin \log x).$

$$P.\,I. = \frac{1}{(\theta+1)(\theta^2-2\theta+2)} \cdot 10\left(x + \frac{1}{x}\right)$$

$$= \frac{1}{(\theta+1)(\theta^2-2\theta+2)} \cdot 10x + \frac{1}{(\theta+1)(\theta^2-2\theta+2)}\, 10x^{-1}$$

$$= \frac{10}{2}\, x + \frac{10}{5\,(\theta+1)}\, x^{-1}$$

$$= 5x + 2x^{-1} \int x^{1-1} \cdot x^{-1}\, dx = 5x + 2x^{-1} \log x.$$

Therefore the solution is

$$y = x\,(c_2 \cos \log x + c_3 \sin \log x + 5) + x^{-1}\,(c_1 + 2 \log x)$$

Ex. 4. *Solve :* $x^3 \frac{d^3y}{dx^3} + 6x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - 4y = 0 \qquad \ldots\ldots(1)$

Substituting $y = x^m$ in (1) we get

$$[\, m(m-1)(m-2) + 6m(m-1) + 4m - 4 \,]\, x^m = 0$$

or $\quad (m^3 + 3m^2 - 4)\, x^m = 0$

or $\quad (m-1)(m+2)^2\, x^m = 0$

Substitution of $y = x^m$ will satisfy the equation (1) if

$$(m-1)(m+2)^2 = 0.$$

The roots of this equation are $1, -2, -2$.

Hence the solution is

$$y = c_1 x + (c_2 + c_3 \log x)\, x^{-2}.$$

Ex. 5. *Solve :* $(x^3 D^3 + 3x^2 D^2 + xD + 1)\, y = 0. \qquad \ldots\ldots(1)$

Substituting $y = x^m$ in (1) we get

$$(m^3 + 1)\, x^m = 0\,;$$

Substitution of $y = x^m$ will satisfy the equation (1)

$$m^3 + 1 = 0$$

of which the roots are $-1,\ \frac{1}{2}(1 \pm \sqrt{3}i)$.

Ex. 9. *Solve* : $(2x-1)^3 \frac{d^3y}{dx^3}+(2x-1)\frac{dy}{dx}-2y=0$ ......(1)

Let $2x-1=z$, then $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=2\frac{dy}{dz}$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(2\frac{dy}{dz}\right)=\frac{d}{dz}\left(2\frac{dy}{dz}\right)\frac{dz}{dx}=4\frac{d^2y}{dz^2}$$

$$\frac{d^3y}{dx^3}=\frac{d}{dx}\left(4\frac{d^2y}{dz^2}\right)=\frac{d}{dz}\left(4\frac{d^2y}{dz^2}\right)\frac{dz}{dx}=8\frac{d^3y}{dz^3}.$$

ostituting these values in the equation (1), it is reduced to

$$8z^3\frac{d^3y}{dz^3}+2z\frac{dy}{dz}-2y=0 \quad \text{......(2)}$$

Putting $y=z^m$, we get

$$\{8m(m-1)(m-2)+2m-2\}z^m=0$$

$$(4m^3-12m^2+9m-1)z^m=0$$

l the roots of the equation are

$1, \frac{1}{2}(2+\sqrt{3})$ and $\frac{1}{2}(2-\sqrt{3})$.

Hence the solution of (2) is

$$y=c_1 z+c_2 z^{1+\frac{1}{2}\sqrt{3}}+c_3 z^{1-\frac{1}{2}\sqrt{3}}$$

l therefore the solution of (1) is

$$y=c_1(2x-1)+c_2(2x-1)^{1+\frac{1}{2}\sqrt{3}}+c_3(2x-1)^{1-\frac{1}{2}\sqrt{3}}$$

$$=(2x-1)\left\{c_1+c_2(2x-1)^{\frac{1}{2}\sqrt{3}}+c_3(2x-1)^{-\frac{1}{2}\sqrt{3}}\right\}$$

Ex. 12. *Solve* $(1+x)^2 y_2+(1+x)y_1+y=4\cos\log(1+x)$ (*Agra* 1950)

Putting $1+x=e^z$ and denoting $\frac{d}{dz}$ by the symbol D, the differential equation is reduced to

$$D(D-1)y+Dy+y=4\cos z.$$

or $$(D^2+1)y=4\cos z.$$

The auxiliary equation is

$$m^2+1=0$$

$\therefore$ $C.F.=c_1\cos z+c_2\sin z=c_1\cos\log(1+x)+c_2\sin\log(1+x)$

$$P.I.=4\frac{1}{D^2+1}\cos z=4.\tfrac{1}{2}z\sin z$$

$$=2\log(1+x)\sin\log(1+x).$$

$\therefore$ The complete solution is

$$y=c_1\cos\log(1+x)+c_2\sin\log(1+x)+2\log(1+x)\sin\log(1+x)$$

Ex. 13. *Solve* :

$$2x^2 y \frac{d^2y}{dx^2} + 4y^2 = x^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx},$$

fter making it homogeneous by the substitution $y=z^2$.

Since $y=z^2$, therefore $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=2z\frac{dz}{dx}$.

$$\frac{d^2y}{dx^2}=2\left(\frac{dz}{dx}\right)^2+2z\frac{d^2z}{dx^2}.$$

Substituting these values in the given equation it is reduced to

$$x^2\frac{d^2z}{dx^2}-x\frac{dz}{dx}+z=0$$

which is a homogeneous linear equation. In symbolic notation it is

$$(\theta^2-2\theta+1)z=0.$$

Hence its solution is

$$z=(c_1+c_2\log x)\,x$$

or

$$y=(c_1+c_2\log x)^2\,x^2.$$

Ex. 15. *Solve* :

$$x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+y=\frac{\log x.\sin(\log x)+1}{x}.$$ (*Agra* 1939, 1962)

On changing the independent variable by putting $x=e^z$, this equation becomes

$$(D^2-4D+1)\,y=e^{-z}.\,z\sin z+e^{-z}$$

The complementary function

$$=c_1\,e^{(2+\sqrt{3})z}+c_2e^{(2-\sqrt{3})z}=c_1\,x^{2+\sqrt{3}}+c_2\,x^{2-\sqrt{3}}.$$

$$=x^2\left(c_1\,x^{\sqrt{3}}+c_2\,x^{-\sqrt{3}}\right)$$

and Particular Integral

$$=\frac{1}{D^2-4D+1}\,e^{-z}.\,z\sin z+\frac{1}{D^2-4D+1}\,e^{-z}$$

$$=e^{-z}.\,\frac{1}{D^2-6D+6}\,z\sin z+\frac{1}{6}\,e^{-z}.$$

$$=e^{-z}\left\{z\cdot\frac{1}{D^2-6D+6}\sin z-\frac{2D-6}{(D^2-6D+6)^2}\sin z\right\}+\frac{1}{6}\,e^{-z}$$

$$=e^{-z}\left\{z\cdot\frac{1}{5-6D}\sin z-\frac{2D-6}{(5-6D)^2}\sin z\right\}+\frac{1}{6}\,e^{-z}$$

$$=e^{-z}\left\{z\,.\frac{5+6D}{61}\sin z+\frac{2D-6}{11+60D}\sin z\right\}+\frac{1}{6}\,e^{-z}$$

Ex. 2. *Solve* :

$$(x^3-x)\,\frac{d^3y}{dx^3}+(8x^2-3)\,\frac{d^2y}{dx^2}+14x\,\frac{dy}{dx}+4y=\frac{2}{x^3}.$$

(*Agra* 1946, *Rajasthan* 1949, 1959)

Here $P_0=x^3-x$, $P_1=8x^2-3$, $P_2=14x$, $P_3=4$. Writing the condition of being exact $4-14+16-6=0$.

Hence the equation is exact. Integration gives

$$(x^3-x)\,\frac{d^2y}{dx^2}+(5x^2-2)\,\frac{dy}{dx}+4xy=-\frac{1}{x^2}+c.$$

Again, on applying the condition of being exact

$$4x-10x+6x=0\ ;$$

hence it is also exact and its integral is

$$(x^3-x)\,\frac{dy}{dx}+(2x^2-1)\,y=\frac{1}{x}+c_1\,x+c_2\ ;$$

This equation is not exact, for $2x^2-1-(3x^2-1)$ is not equal to zero ; but it is a linear equation of the first order *i.e.*

$$\frac{dy}{dx}+\frac{2x^2-1}{x^3-x}\,y=\frac{1}{x(x^3-x)}+\frac{c_1}{x^2-1}+\frac{c_2}{x^3-x}$$

which on integration gives

$$xy\,\sqrt{(x^2-1)}=\sec^{-1}x+c_1\,\sqrt{(x^2-1)}+c_2\log\left[x+\sqrt{(x^2-1)}\right]+c_3.$$

Ex. 3. Solve :

Ex. 3. *Solve* :

$$\frac{d^3y}{dx^3}+\cos x\,\frac{d^2y}{dx^2}-2\sin x\,\frac{dy}{dx}-y\cos x=\sin 2x$$

(*Allahabad* 1953, *Ag*

It satisfies the condition of being exact *i.e.*

$-\cos x+2\cos x-\cos x=0$. Hence it is exact.

Integrating

$$\frac{d^2y}{dx^2}+\cos x\,\frac{dy}{dx}-y\sin x=-\tfrac{1}{2}\cos 2x+c_1$$

Ex. 2. *Solve* : $\sin^2 x \dfrac{d^2y}{dx^2}=2y.$

(*Agra* 1956, 1962, *Jaipur* '49, '52, *Andhra* '53)

Here $y=\cot x$ is a solution of $\dfrac{d^2y}{dx^2}-2y \operatorname{cosec}^2 x=0.$

Substituting $y=v \cot x$ and therefore

$$\frac{dy}{dx}=\cot x \frac{dv}{dx}-v \operatorname{cosec}^2 x$$

and $\dfrac{d^2y}{dx^2}=\cot x \dfrac{d^2v}{dx^2}-2 \operatorname{cosec}^2 x \dfrac{dv}{dx}+2 \operatorname{cosec}^2 x \cot x,$

ie original equation pecomes

$$\frac{d^2v}{dx^2}-\frac{2}{\sin x \cos x}\frac{dv}{dx}=0.$$

Substituting p for $\dfrac{dv}{dx}$ this becomes

The equation is still exact. Hence integrating again

$$\frac{dy}{dx}+y \cos x=-\tfrac{1}{4} \sin 2x+c_1 x+c_2$$

This equation is linear. Hence integrating

$$y\, e^{\sin x}=\int -\tfrac{1}{4} \sin 2x \,.\, \sin x\, dx+\int e^{\sin x}(c_1 x+c_2)\, dx+c_3$$

$$=-\tfrac{1}{2}\int \sin x \cos x\, e^{\sin x} dx+\int e^{\sin x}(c_1 x+c_2)\, dx+c_3$$

Now $\displaystyle\int \sin x \cos x\, e^{\sin x} dx=\sin x\, e^{\sin x}-\int \cos x\, e^{\sin x} dx$

$$=\sin x\, e^{x}-e^{\sin x}$$

$\therefore \quad y\, e^{\sin x}=-\tfrac{1}{2}(\sin x-1)\, e^{\sin x}=\displaystyle\int e^{\sin x}(c_1 x+c_2)\, dx+c_3$

or $\quad y=-\tfrac{1}{2}(\sin x-1)+e^{-\sin x}\displaystyle\int e^{\sin x}(c_1 x+c_2)\, dx+c_3 e^{-\sin x}$

Ex. 3. *Solve :* $(1-x^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=x\,(1-x^2)^{3/2}$

(*I. A. S.* 1951, *Jaipur* 1951, *Punjab*

Here $y=x$ is obviously a solution of the equation

$$(1-x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0.$$

Substituting $y=vx$ and therefore $\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

d $$\frac{d^2y}{dx^2}=x\frac{d^2v}{dx^2}+2\frac{dv}{dx},$$

original equation becomes

$$\frac{d^2v}{dx^2}+\left(\frac{2}{x}+\frac{x}{1-x^2}\right)\frac{dv}{dx}=(1-x^2)^{1/2}.$$

Substituting for $\dfrac{dv}{dx}$ this becomes

$$\frac{dp}{dx}+\left(\frac{2}{x}+\frac{x}{1-x^2}\right)p=(1-x^2)^{1/2}$$

This is linear and the integrating factor is $x^2/(1-x^2)^{1/2}$,

Therefore $p\;\dfrac{x^2}{\sqrt{(1-x^2)}}=\dfrac{1}{3}x^3+c_1$

$$\text{or}\quad p=\frac{dv}{dx}=\frac{1}{3}\,x\sqrt{1-x^2}+\frac{c_1}{x^2}\sqrt{1-x^2}.$$

Ex. 8. *Solve :*

$$x^2\frac{d^2y}{dx^2}-2\,(x^2+x)\frac{dy}{dx}+(x^2+2x+2)\,y=0.$$ (*Agra* 1

The equation may be written in the form

$$\frac{d^2y}{dx^2}-2\left(1+\frac{1}{x}\right)\frac{dy}{dx}+\left(1+\frac{2}{x}+\frac{2}{x^2}\right)y=0.$$

Here $P=-2\left(1+\dfrac{1}{x}\right)$, $Q=1+\dfrac{2}{x}+\dfrac{2}{x^2}$; and hence

$$u=e^{-\frac{1}{2}\int P\,dx}=e^{\int(1+1/x)\,dx}=e^{x+\log x}=xe^x.$$

$$\therefore\quad I=Q-\tfrac{1}{2}\frac{dP}{dx}-\tfrac{1}{4}P^2=0.$$

Hence the transformed equation is

$$\frac{d^2v}{dx^2}=0\quad\text{or}\quad\frac{dv}{dx}=c_1\quad\text{or}\quad v=c_1x+c_2$$

Therefore the general solution of the given equation is

$$y=u\,v=x\,e^x\,(c_1x+c_2)=e^x\,(c_1x^2+c_2x).$$

Ex. 10. *Solve* :

$$\frac{d^2y}{dx^2}+\frac{1}{x^{1/3}}\frac{dy}{dx}+\left(\frac{1}{4x^{2/3}}-\frac{1}{6x^{4/3}}-\frac{6}{x^2}\right)y=0.$$

(*Allahabad* 1942, *Jaipur* 1951

Here $P=x^{-\frac{1}{3}}$, $Q=\frac{1}{4x^{2/3}}-\frac{1}{6x^{4/3}}-\frac{6}{x^2}$; and hence

$$u=e^{-\frac{1}{2}\int P\,dx}=e^{-\frac{3}{4}x^{2/3}}$$

nd $I=\frac{1}{4x^{2/3}}-\frac{1}{6x^{4/3}}-\frac{6}{x^2}+\frac{1}{6}\frac{1}{x^{4/3}}-\frac{1}{4}\frac{1}{x^{2/3}}=-\frac{6}{x^2}.$

Hence the transformed equation is

$$\frac{d^2v}{dx^2}-\frac{6v}{x^2}=0 \quad \text{or} \quad x^2\frac{d^2v}{dx^2}-6v=0.$$

This is a homogeneous linear equation and its solution is

$$v=c_1x^3+c_2x^{-2}.$$

Hence the complete primitive is

$$y=uv=e^{-\frac{3}{4}x^{2/3}}(c_1x^3+c_2'x^{-2}).$$

Ex. 10. *Solve* :

$$\frac{d^2y}{dx^2}+\frac{1}{x^{1/3}}\frac{dy}{dx}+\left(\frac{1}{4x^{2/3}}-\frac{1}{6x^{4/3}}-\frac{6}{x^2}\right)y=0.$$

(*Allahabad* 1942, *Jaipur* 1951

Here $P=x^{-\frac{1}{3}}$, $Q=\frac{1}{4x^{2/3}}-\frac{1}{6x^{4/3}}-\frac{6}{x^2}$; and hence

$$u=e^{-\frac{1}{2}\int P\,dx}=e^{-\frac{3}{4}x^{2/3}}$$

nd $I=\frac{1}{4x^{2/3}}-\frac{1}{6x^{4/3}}-\frac{6}{x^2}+\frac{1}{6}\frac{1}{x^{4/3}}-\frac{1}{4}\frac{1}{x^{2/3}}=-\frac{6}{x^2}.$

Hence the transformed equation is

$$\frac{d^2v}{dx^2}-\frac{6v}{x^2}=0 \quad \text{or} \quad x^2\frac{d^2v}{dx^2}-6v=0.$$

This is a homogeneous linear equation and its solution is

$$v=c_1x^3+c_2x^{-2}.$$

Hence the complete primitive is

$$y=uv=e^{-\frac{3}{4}x^{2/3}}(c_1x^3+c_2'x^{-2}).$$

Ex. 12. *Solve the equation :*

$$\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+(4x^2-1)\,y=-3\;e^{x^2}\sin 2x.$$

(*Agra* 1933, 1941, 1950, 1955 ; *Luck*. 1951, 1956 ; *I.A.S.* 1956)

Here $P=-4x,\ Q=4x^2-1$;

hence $u=e^{-\frac{1}{2}\int P\,dx}=e^{\int 2x\,dx}=e^{x^2}$

$$I=Q-\tfrac{1}{2}\frac{dP}{dx}-\tfrac{1}{4}P^2=4x^2-1-\tfrac{1}{2}(-4)-\tfrac{1}{4}\cdot 16x^2=1.$$

Hence the transformed equation is

$$\frac{d^2v}{dx^2}+v=-3\;e^{x^2}\sin 2x\;e^{\frac{1}{2}\int -4x\,dx}=-3\sin 2x.$$

The solution of this equation is

$$v=c_1\cos x+c_2\sin x+\sin 2x.$$

Hence the complete primitive is

$$y=vu=e^{x^2}(c_1\cos x+c_2\sin x+\sin 2x).$$

Ex. 22. Solve : $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=x^2e^x.$

(*Agra* 1935, 1943, 1960 ; *I. A. S.* 1954)

Arranging in the standard form, this is

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=e^x.$$

Particular solutions of the equation when the right-hand member is zero are $u=x$ and $v=x^{-1}$.

Let the primitive be

$$y=Ax+Bx^{-1}.$$

Two equations to connect A and B are

$$A_1x+B_1x^{-1}=0 \qquad \text{...(1)}$$

and

$$A_1-\frac{1}{x^2}B_1=e^x \qquad \text{...(2)}$$

Solving (1) and (2) $A_1=\frac{1}{2}e^x$, whence $A=\frac{1}{2}e^x+a$

and $B_1=-\frac{1}{2}e^x.x^2$, whence $B=-\frac{1}{2}e^x x^2+x\,e^x-e^x+b$.

Hence the complete primitive is

$$y=\tfrac{1}{2}x\,e^x+a\,x-\tfrac{1}{2}e^x.x+e^x-e^x x^{-1}+b\,x^{-1}.$$
$$=a\,x+b\,x^{-1}+e^x-e^x x^{-1}.$$

Ex. 24. *Solve :* $\frac{d^2y}{dx^2}+(1-\cot x)\frac{dy}{dx}-y\cot x=\sin^2 x.$

(*Jaipur* 1956, 1958)

Let the left hand side equated to zero be solved.

Obviously $y=e^{-x}$ is a solution of it.

Let $y=v\,e^{-x}$. Then the is equation is reduced to

$$\frac{d^2v}{dx^2}-(1+\cot x)\frac{dv}{dx}=0$$

which on integration gives

$$c_1\,v=-\tfrac{1}{2}e^x(\cos x-\sin x)+c_2$$
$$y=vy_1=A(\cos x-\sin x)+Be^{-x}$$

$\therefore$ The C F. of the given equation is

$$y=A(\cos x-\sin x)+Be^{-x}$$

Two equations to connect A and B are

$$A_1(\cos x-\sin x)+B_1\,e^{-x}=0 \qquad \ldots(1)$$

and $$A_1(-\sin x-\cos x)-B_1\,e^{-x}=\sin^2 x \qquad \ldots(2)$$

Solving (1) and (2) $A_1=-\frac{1}{2}\sin x$ whence $A=\frac{1}{2}\cos x+c_1$

and $B_1=\frac{1}{4}e^x\sin 2x-\frac{1}{4}e^x(1-\cos 2x)$

whence $B=-\frac{1}{20}e^x(2\cos 2x-\sin 2x)-\frac{1}{4}e^x$
$\qquad -\frac{1}{20}e^x(-2\sin 2x-\cos 2x)+c_2$

$\therefore$ Complete solution is

$$y=A(\cos x-\sin x)+B\,e^{-x}$$
$$=c_1(\cos x-\sin x)-\tfrac{1}{10}(\sin^2 2x-2\cos 2x)+c_2\,e^{-x}.$$

Ex 25. *Solve :* $y_3-6y_2+11y_1-6y=e^{2x}$. (***Agra* 1939, 1962**)

Particular solutions of the equation when the right hand member is zero are $u=e^x$, $v=e^{2x}$, $w=e^{3x}$.

Let the primitive be

$$y=A\,e^x+B\,e^{2x}+C\,e^{3x}.$$

Three equations to connect A, B and C are

$$e^x A_1+e^{2x}B_1+e^{3x}C_1=0,$$
$$e^x A_1+2e^{2x}B_1+3\,e^{3x}C_1=0 \qquad \ldots(2)$$
$$e^x A_1+4\,e^{2x}B_1+9\,e^{3x}C_1=e^{2x}. \qquad \ldots(3)$$

From (1) and (2)

$$\frac{A_1}{e^{5x}}=\frac{B_1}{-2e^{4x}}=\frac{C_1}{e^{3x}}=\lambda \text{ (say)}$$

Substituting the values of A_1, B_1, C_1 in (3)

$$e^{2x}=\lambda(e^{6x}-8e^{6x}+9e^{6x})=\lambda\,2e^{6x}$$

Hence $\lambda=\frac{1}{2}e^{-4x}$.

$$A_1=\tfrac{1}{2}e^x,\ B_1=-1,\ C_1=\tfrac{1}{2}e^{-x}$$

Integrating $A=\frac{1}{2}e^x+a$, $B=-x+b$, $C=-\frac{1}{2}e^{-x}+c$

The complete primitive, therefore is

$$y=\tfrac{1}{2}e^{2x}+a\,e^x-x\,e^{2x}+be^{2x}-\tfrac{1}{2}e^{2x}+ce^{3x}$$
$$=a\,e^x+b\,e^{2x}+c\,e^{3x}-x\,e^{2x}.$$

Example 22 Solve the differential equation

$$(\tan^{-1}y - x)\, dy = (1 + y^2)\, dx.$$

Solution The given differential equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

Now (1) is a linear differential equation of the form $\frac{dx}{dy} + P_1\, x = Q_1$,

where, $P_1 = \frac{1}{1+y^2}$ and $Q_1 = \frac{\tan^{-1}y}{1+y^2}$.

Therefore, $\text{I.F} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$

Thus, the solution of the given differential equation is

$$x e^{\tan^{-1} y} = \int \left(\frac{\tan^{-1}y}{1+y^2}\right) e^{\tan^{-1} y}\, dy + C$$

Let $$I = \int \left(\frac{\tan^{-1}y}{1+y^2}\right) e^{\tan^{-1} y}\, dy$$

Substituting $\tan^{-1} y = t$ so that $\left(\frac{1}{1+y^2}\right) dy = dt$, we get

$$I = \int t\, e^t dt = t\, e^t - \int 1\, .\, e^t\, dt = t\, e^t - e^t = e^t\,(t - 1)$$

or $$I = e^{\tan^{-1} y}(\tan^{-1}y - 1)$$

Substituting the value of I in equation (2), we get

$$x\, .\, e^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + C$$

or $$x = (\tan^{-1}y - 1) + C\, e^{-\tan^{-1} y}$$

Previous Year Question Paper III sem -2024

Part-A/भाग-अ

[10×2=20]

1. Answer *all* parts of the question:

प्रश्न के सभी खण्डों के उत्तर दीजिए:

(a) Show that every open interval]a, b[is a neighbourhood of each of its points.

प्रदर्शित कीजिए कि प्रत्येक विवृत्त अंतराल अपने प्रत्येक बिन्दु का प्रतिवेश होता है।

(b) Find the limit points of the set

$$S = \left\{ \pm 1, \pm \frac{3}{2}, \pm \frac{4}{3}, \ldots\ldots\ldots \right\}.$$

समुच्चय $S = \left\{ \pm 1, \pm \frac{3}{2}, \pm \frac{4}{3}, \ldots\ldots\ldots \right\}$. के सीमा बिन्दु ज्ञात कीजिए।

(c) If $I_n = \left]x - \frac{1}{n}, x + \frac{1}{n}\right[$, $n \in N$, then show that $\bigcap_{n=1}^{\infty} I_n$ is not an open set.

यदि $I_n = \left]x - \frac{1}{n}, x + \frac{1}{n}\right[$, $n \in N$, तब प्रदर्शित कीजिए कि $\bigcap_{n=1}^{\infty} I_n$ एक विवृत्त समुच्चय नहीं है।

(d) Find the limit points of the sequence $\left\langle (-1)^n \left(1 + \frac{1}{n}\right) \right\rangle$.

अनुक्रम $\left\langle (-1)^n \left(1 + \frac{1}{n}\right) \right\rangle$ के सीमा बिन्दु ज्ञात कीजिए।

(e) Show that the sequence $\left\langle \frac{1}{n} \right\rangle$ is a Cauchy sequence.

प्रदर्शित कीजिए कि अनुक्रम $\left\langle \frac{1}{n} \right\rangle$ एक कौशी अनुक्रम है।

(f) Show that the function $f(x)=\begin{cases} x, \text{ if } x \text{ is rational} \\ 1-x, \text{ if } x \text{ is irrational} \end{cases}$ is continuous at $x=\frac{1}{2}$.

प्रदर्शित कीजिए कि फलन $f(x)=\begin{cases} x, \text{ if } x \text{ is rational} \\ 1-x, \text{ if } x \text{ is irrational} \end{cases}$ बिन्दु $x=\frac{1}{2}$ पर संतत् है।

(g) Solve the following differential equation:

निम्नलिखित अवकल समीकरण को हल कीजिए:

$$ydx + xdy + \frac{xdy - ydx}{xy} = 0$$

(h) Find the particular integral of the differential equation $(D^2+4D-12)y = e^{2x}$

अवकल समीकरण $(D^2+4D-12)y = e^{2x}$ का विशिष्ट समाकल ज्ञात कीजिए।

(i) Find the complementary function of the homogeneous differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2$$

समघात अवकल समीकरण $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2$ का पूरक फलन ज्ञात कीजिए।

(j) Find one of the part of the complementary function of the following differential equation:

$$x^2\frac{d^2y}{dx^2} - (x^2-2x)\frac{dy}{dx} + (x+2)y = x^3e^x$$

निम्नलिखित अवकल समीकरण के पूरक फलन का एक भाग ज्ञात कीजिए।

$$x^2\frac{d^2y}{dx^2} - (x^2-2x)\frac{dy}{dx} + (x+2)y = x^3e^x$$

Part-B/भाग-ब

[2×10=20]

Attempt any two questions out of the following four questions.

निम्नलिखित चार प्रश्नों में से किन्हीं दो प्रश्नों के उत्तर दीजिए।

2. Prove that every bounded and infinite set has atleast one limit point.

   सिद्ध कीजिए कि प्रत्येक परिबद्ध एवं अपरिमित समुच्चय का कम से कम एक सीमा बिन्दु अवश्य होता है।

3. Prove that the necessary and sufficient condition for a sequence $\langle x_n \rangle$ to be a convergent sequence is that $\langle x_n \rangle$ is bounded and has a unique limit point.

   सिद्ध कीजिए कि अनुक्रम $\langle x_n \rangle$ को एक अभिसारी अनुक्रम होने के लिए आवश्यक और पर्याप्त प्रतिबन्ध है कि $\langle x_n \rangle$ परिबद्ध है तथा इसका अद्वितीय सीमा बिन्दु है।

4. Solve the following differential equation:

   $$(x^3y^3 + x^2y^2 + xy + 1)y\,dx + (x^3y^3 - x^2y^2 - xy + 1)x\,dy = 0$$

   निम्न अवकल समीकरण को हल कीजिए।

   $$(x^3y^3 + x^2y^2 + xy + 1)y\,dx + (x^3y^3 - x^2y^2 - xy - 1)x\,dy = 0$$

5. Solve the following differential equation:

   $$x^4\frac{d^3y}{dx^3} + 2x^3\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 1$$

   निम्न अवकल समीकरण को हल कीजिए।

   $$x^4\frac{d^3y}{dx^3} + 2x^3\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 1.$$

Attempt all questions. [4×20=80]

सभी प्रश्नों के उत्तर दीजिए।

6. (a) Prove that the intersection of a finite number of open sets is an open set.

सिद्ध कीजिए कि परिमित संख्या में विवृत समुच्चयों का सर्वनिष्ठ निर्धारण एक विवृत समुच्चय होता है।

(b) Prove that a set is closed if and only if its complement is open.

सिद्ध कीजिए कि कोई समुच्चय एक संवृत समुच्चय होता है यदि और केवल यदि इसका पूरक समुच्चय एक विवृत समुच्चय है।

OR/अथवा

(a) Prove that the derived set of any set is a closed set.

सिद्ध कीजिए कि किसी भी समुच्चय का व्युत्पन्न समुच्चय सदैव एक संवृत समुच्चय होता है।

(b) Prove that the set R of real numbers is not a compact set.

सिद्ध कीजिए कि वास्तविक संख्याओं का समुच्चय R एक संहत समुच्चय नहीं है

7. (a) Prove that the necessary and sufficient condition for a monotonically increasing sequence $\langle x_n \rangle$ to be convergent is that it is bounded and in that case $\lim x_n = Sup\{\langle x_n \rangle\}$.

सिद्ध कीजिए कि एकदिष्ट वर्धमान अनुक्रम $\langle x_n \rangle$ के अभिसरण के लिए आवश्यक एवं पर्याप्त प्रतिबन्ध है कि $\langle x_n \rangle$ परिबद्ध है तथा इस अवस्था में $\lim x_n = Sup\{\langle x_n \rangle\}$.

(b) Prove that the sequence $\langle x_n \rangle$, where $x_1 = \frac{1}{2}$ and $x_{n+1} = \frac{2x_n + 1}{3}$, $n \in N$ is convergent. Also find its limit.

सिद्ध कीजिए कि अनुक्रम $\langle x_n \rangle$, जहाँ $x_1 = \frac{1}{2}$ तथा $x_{n+1} = \frac{2x_n + 1}{3}$, $n \in N$ एक अभिसारी अनुक्रम है। इसकी सीमा भी ज्ञात कीजिए।

OR/अथवा

(a) Prove that the necessary and sufficient condition for the $\langle x_n \rangle$ to be a convergent sequence is that it is a Cauchy sequence.

सिद्ध कीजिए कि अनुक्रम $\langle x_n \rangle$ एक अभिसारी अनुक्रम है यदि और केवल यदि $\langle x_n \rangle$ एक कौशी अनुक्रम है।

(b) Show that the sequence $\left\langle 1/n^2 \right\rangle$ is a Cauchy sequence.

प्रदर्शित कीजिए कि अनुक्रम $\left\langle 1/n^2 \right\rangle$ एक कौशी अनुक्रम है

8. (a) Solve the following differential equation:

निम्न अवकल समीकरण को हल कीजिए :

$$\left(xy^2 - x^2\right)dx + (3x^2y^2 + x^2y - 2x^3 + y^2)dy = 0$$

(b) Solve the following differential equation:

निम्न अवकल समीकरण को हल कीजिए :

$$p^3 - 4xyp + 8y^2 = 0$$

OR/अथवा

(a) Solve the following linear differential equation:

निम्न रैखिक अवकल समीकरण का हल ज्ञात कीजिए :

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = x \cos x$$

(b) Solve the following differential equation:

निम्नलिखित अवकल समीकरण का हल ज्ञात कीजिए:

$$(D^2 - 9D + 18)y = e^{e^{-3x}}$$

9. (a) Solve the following differential equation:

$$\frac{d^2y}{dx^2} + (1 - \cot x)\frac{dy}{dx} - y \cot x = \sin^2 x$$

निम्नलिखित अवकल समीकरण को हल कीजिए:

$$\frac{d^2y}{dx^2} + (1 - \cot x)\frac{dy}{dx} - y \cot x = \sin^2 x$$

(b) Solve the following differential equation:

$$x\frac{d}{dx}\left(x\frac{dy}{dx} - y\right) - 2x\frac{dy}{dx} + 2y + x^2y = 0$$

निम्नलिखित अवकल समीकरण को हल कीजिए:

$$x\frac{d}{dx}\left(x\frac{dy}{dx} - y\right) - 2x\frac{dy}{dx} + 2y + x^2y = 0$$

OR/अथवा

(a) Solve the following differential equation:

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 4x^3 \sin x^2$$

निम्नलिखित अवकल समीकरण को हल कीजिए:

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 4x^3 \sin x^2$$

(bi) Solve the following differential equation using method of variation of parameters:

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x^2e^x$$

फलन विधि का प्रयोग करते हुए निम्न अवकल समीकरण का हल ज्ञात कीजिए:

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x^2e^x$$
